PONTRJAGIN-THOM MAPS AND THE HOMOLOGY OF THE MODULI STACK OF STABLE CURVES

JOHANNES EBERT AND JEFFREY GIANSIRACUSA

ABSTRACT. We study the singular homology (with field coefficients) of the moduli stack $\overline{\mathfrak{M}}_{g,n}$ of stable n-pointed complex curves of genus g. Each irreducible boundary component of $\overline{\mathfrak{M}}_{g,n}$ determines via the Pontrjagin-Thom construction a map from $\overline{\mathfrak{M}}_{g,n}$ to a certain infinite loop space whose homology is well understood. We show that these maps are surjective on homology in a range of degrees proportional to the genus. This detects many new torsion classes in the homology of $\overline{\mathfrak{M}}_{g,n}$.

CONTENTS

1. INTRODUCTION

Let $\overline{\mathfrak{M}}_{g,n}$ denote the moduli stack of stable nodal complex curves of genus g with n labeled marked points; this is the Deligne-Mumford-Knudsen compactification of the moduli stack $\mathfrak{M}_{g,n}$ of smooth curves. This object plays a central role in Gromov-Witten theory, conformal field theory, and conjecturally in string topology. The rational cohomology of $\overline{\mathfrak{M}}_{g,n}$ and its tautological subalgebra have been extensively studied in the literature, and the structure of the tautological algebra is at least conjecturally known. However, the mod p (co)homology has received relatively little attention. Here the distinction between the *moduli stack* and the associated *coarse moduli space* becomes important because they are only *rationally* homology isomorphic. We take the point of view that the moduli stack is the more fundamental object.

Using the proof of the integrally refined Mumford conjecture by Madsen and Weiss [MW07], Galatius [Gal04] completely computed the mod p homology of $\mathfrak{M}_{g,n}$ in the Harer-Ivanov stable range; there are large families of torsion classes. Here we address the question of torsion in the homology of the compactified moduli stack.

The boundary $\partial\overline{\mathfrak{M}}_{g,n} := \overline{\mathfrak{M}}_{g,n} \setminus \mathfrak{M}_{g,n}$ of $\overline{\mathfrak{M}}_{g,n}$ is a union of substacks of complex codimension 1. These irreducible boundary components are the images of the 'gluing' morphisms between moduli stacks defined by identifying two marked points together to form a node. Let P be a subset of $\{1,\dots,n\}$. The gluing morphisms are:

$$\begin{aligned} \xi_{irr} &: \overline{\mathfrak{M}}_{g-1,n+(2)} \to \overline{\mathfrak{M}}_{g,n}, \\ \xi_{h,P} &: \overline{\mathfrak{M}}_{h,P\sqcup\{p_1\}} \times \overline{\mathfrak{M}}_{g-h,P^c\sqcup\{p_2\}} \to \overline{\mathfrak{M}}_{g,n}, \end{aligned} \tag{1.1}$$

where $\overline{\mathfrak{M}}_{g-1,n+(2)}$ is the moduli stack of stable curves with $n+2$ marked points, the first n of which are labeled. These morphisms are representable proper immersions (in fact embeddings when P is a proper subset) of complex codimension 1 with transversal (self-)intersections and their images are precisely the various irreducible components of the boundary.

We study the effect on homology of the Pontrjagin-Thom maps for these immersions. We show that the (self)-intersections produce large families of torsion homology classes which are unrelated to the known torsion classes on $\mathfrak{M}_{g,n}$.

Recall that if $N^{n-k} \looparrowright M^n$ is a proper immersion of real codimension k between smooth manifolds then the classical Pontrjagin-Thom construction produces a map $M \to QN^{\nu(f)}$, where $QX = \Omega^\infty\Sigma^\infty X$ is the free infinite loop space generated by X, and $N^{\nu(f)}$ is the Thom space of the normal bundle $\nu(f)$. A reduction of the structural group of $\nu(f)$ to $G \xrightarrow{j} GL_k(\mathbb{R})$ induces a map

$$QN^{\nu(f)} \to QBG^{j^*\gamma_k},$$

where γ_k is the universal k-plane bundle over $BGL_k(\mathbb{R})$. Thus we obtain a map

$$M \to QBG^{j^*\gamma_k}.$$

In section 3 we extend the classical construction of Pontrjagin-Thom maps to the category of differentiable local quotient stacks. A stack $\mathfrak{X}$ admitting an atlas has an associated *homotopy type* $\mathrm{Ho}(\mathfrak{X})$ (see section 2) which is a space that has the same homological invariants as the stack, and the Pontrjagin-Thom construction produces a map out of the homotopy type.

Let $T(2) = U(1)\times U(1)$ denote the maximal torus in $U(2)$, and let $N(2) \cong U(1) \wr \mathbb{Z}/2 = U(1)^2 \rtimes \mathbb{Z}/2$ denote the normalizer of the maximal torus. There are homomorphisms

$$T(2) \hookrightarrow N(2) \to U(1),$$

where the first arrow is the inclusion and the second is defined by multiplying the $U(1)$ components together; we write V for the universal line bundle over $BU(1)$ or its pullback to $BN(2)$ or $BT(2)$. The normal bundle of ξ_{irr} comes with a reduction of structure group to $N(2)$, and the structure group of the normal bundle of $\xi_{h,P}$ reduces to $T(2)$. Thus we have Pontrjagin-Thom maps

$$\begin{aligned} \Phi_{irr} &: \mathrm{Ho}(\overline{\mathfrak{M}}_{g,n}) \to QBN(2)^V, \\ \Phi^1_{h,P} &: \mathrm{Ho}(\overline{\mathfrak{M}}_{g,n}) \to QBT(2)^V, \\ \Phi^0_{h,P} &: \mathrm{Ho}(\overline{\mathfrak{M}}_{g,n}) \to QBT(2)^V \to QBU(1)^V, \end{aligned}$$

where $\Phi^0_{h,P}$ is the composition of $\Phi^1_{h,P}$ with the map induced by the multiplication $T(2) \to U(1)$. Our main theorem is the following.

Theorem 1.2. *Let g and n be fixed. Let $\mathbb{F}$ be a field (of arbitrary characteristic).*

(1) *The map Φ_{irr} is surjective on $H_i(-,\mathbb{F})$ for $i \leq (g-2)/4$.*
(2) *The map $\Phi^1_{h,\emptyset}$ is surjective on $H_i(-,\mathbb{F})$ for $i \leq (h/2-1)$, $i \leq (g-2)/(2h+2)$.*
(3) *The map $\Phi^0_{h,\emptyset}$ is surjective on $H_i(-,\mathbb{F})$ for $i \leq (g-2)/(2h+2)$.*

This theorem detects large families of new torsion classes in the (co)homology of $\overline{\mathfrak{M}}_{g,n}$ as follows. Let Φ be one of the above maps. On cohomology with field coefficients the induced map Φ^* is injective in one of the above ranges of degrees.

Rationally. The cohomology of $QBN(2)^V$ with coefficients in $\mathbb{Q}$ is the free commutative algebra on generators $a_{i,j}$ ($i,j \geq 0$) of degree $2+2i+4j$. In this case the image of Φ^* is contained in the tautological algebra; see section 6.

Mod p. The mod p Betti numbers of $QBN(2)^V$ are much larger than the rational Betti numbers. If $\mathrm{char}(\mathbb{F}) > 0$, then $H_*(QBN(2)^V;\mathbb{F})$ has a large and rich structure — it is the free graded-commutative algebra over the free Dyer-Lashof module generated by $\widetilde{H}_*(BN(2)^V;\mathbb{F})$; see sections 5.5 and 5.7 for details. Hence this detects large families of new mod p cohomology classes of $\overline{\mathfrak{M}}_{g,n}$ which are *not* reductions of rationally nontrivial classes.

Remark 1.3. (1) More generally, one can take the cartesian product of several of these Pontrjagin-Thom maps and the induced map on homology will be surjective in a range of degrees. However, stating the exact range of degrees becomes somewhat cumbersome. The more general result is Theorem 5.1.
(2) Note that the range of surjectivity is proportional to g in (1) and (3) but not in (2). On the other hand, the homology groups of the target in (2) are somewhat larger than those of the target in (3), so $\Phi^1_{h,\emptyset}$ detects more classes than $\Phi^0_{h,\emptyset}$ but in a reduced range of degrees.
(3) When $\emptyset \neq P \subsetneq \{1,\dots,n\}$ the morphism $\xi_{h,P}$ is an embedding. Therefore its Pontrjagin-Thom map factors through $BT(2)^V \to QBT(2)^V$. The cohomology classes pulled back from $BT(2)^V$ all lie in the tautological ring of $\overline{\mathfrak{M}}_{g,n}$. However, one can also consider the quotient $\overline{\mathfrak{M}}_{g,n}/\!\!/\Sigma_n$, where the symmetric group acts by permuting the labels of the marked points. Now the gluing morphism
$$\xi_{h,P} : \overline{\mathfrak{M}}_{h,P\sqcup\{p_1\}}/\!\!/\Sigma_P \times \overline{\mathfrak{M}}_{g-h,P^c\sqcup\{p_2\}}/\!\!/\Sigma_{P^c} \to \overline{\mathfrak{M}}_{g,n}/\!\!/\Sigma_n$$
is an immersion with nontrivial self-intersections whenever $h < g/2$. In this case one can easily adapt the proof of Theorem 1.2 to show for instance that the associated Pontrjagin-Thom map $\Phi^0_{h,P}$ is surjective on homology in degrees $i \leq (g-2)/(2h+2)$, provided that $n \geq |P|(g-2)/(2h+2)$.
(4) Finally we mention that the restriction of the Pontrjagin-Thom maps to the moduli stack $\mathfrak{M}_{g,n}$ of smooth curves is nullhomotopic, because the images of the natural morphisms (1.1) lie in $\partial\overline{\mathfrak{M}}_{g,n}$. Thus the torsion classes we detect are *not* related to the torsion classes on $\mathfrak{M}_{g,n}$ which were computed by Galatius [Gal04].

There is a certain overlap between Theorem 1.2 and unpublished work by Eliashberg and Galatius [GE06]. They announced a determination of the homotopy type of the moduli stack of stable irreducible curves as the genus tends to infinity. Their result should imply our theorem for the Pontrjagin-Thom map Φ_{irr}. However, they do not consider the other boundary strata.

Outline. In section 2 we recall some material on stacks and explain the notion of the *homotopy type* of a topological stack. In section 3, we show how to generalize the Pontrjagin-Thom construction to proper morphisms of local quotient stacks. Section 4 reviews some needed facts about the geometry of the moduli stack $\overline{\mathfrak{M}}_{g,n}$. In section 5 we state our main theorem in full generality and prove it. In section 6 we describe how the classes we detect rationally are related to the tautological algebra.

Acknowledgements. The first author was supported by a postdoctoral grant from the German Academic Exchange Service (DAAD). Some preliminary parts of this work were done during his stay at the Sonderforschungsbereich "Geometrische Strukturen in der Mathematik" at the Mathematical Institute in Münster; he thanks Wolfgang Lück for his invitation. The second author thanks the IHES for its hospitality and C.F. Bödigheimer for an invitation to Bonn, where this project was begun. Both authors thank Ulrike Tillmann for helpful comments.

2. Some homotopy theory for topological stacks

In this section we set up the homotopical framework in which the Pontrjagin-Thom maps for stacks will reside.

2.1. **Generalities on stacks.** We will assume that the reader is comfortable with the language of stacks and therefore we will not repeat the basic definitions in detail. A stack over a site **S** is a lax sheaf of groupoids over **S**. We will consider the sites **diff** and **top** of smooth manifolds and topological spaces. The reader is referred to [Hei05] and [Noo05b] for readable introductions to the theory of stacks over the sites **diff** and **top**.

On the site **diff** there is a subtlety in the definition of representable morphisms since one needs transversality for the pullback of two smooth maps to be a smooth manifold. We propose a definition which differs slightly from that given in [Hei05].

Definition 2.1.
(1) A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of stacks on the site **diff** is a *representable submersion* if for any manifold M and any morphism $M \to \mathfrak{X}$, the fiber product $M \times_{\mathfrak{Y}} \mathfrak{X}$ is a smooth manifold and the induced map $M \times_{\mathfrak{Y}} \mathfrak{X} \to M$ is a submersion.
(2) A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of stacks over **S** is *representable* if for any representable submersion $f : M \to \mathfrak{Y}$, the pullback $M \times_{\mathfrak{Y}} \mathfrak{X}$ is a smooth manifold and the induced map $M \times_{\mathfrak{Y}} \mathfrak{X} \to M$ is a smooth map.

With this definition any smooth map between manifolds is representable when considered as a morphism of stacks and any morphism from a smooth manifold to a stack over **diff** is representable. Let $\mathfrak{X}$ be a stack over **diff**. An *atlas* is a smooth manifold X together with a representable submersion $p : X \to \mathfrak{X}$. A stack which admits an atlas is called a *differentiable stack*.

Similarly, we can define topological stacks. We say that a representable morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of stacks over **top** *has local sections* if for any space Y and any map $Y \to \mathfrak{Y}$, the pullback $\mathfrak{X} \times_{\mathfrak{Y}} Y \to Y$ admits local sections (observe that maps which have local sections are surjective and having local sections is a property which is invariant under base-change). An atlas for a stack $\mathfrak{X}$ over **top** is a space X together with a representable

morphism $p : X \to \mathfrak{X}$ having local sections. A *topological stack* is a stack $\mathfrak{X}$ over **top** which admits an atlas. Our terminology differs from that used by Noohi [Noo05b]: the topological stacks defined above are called "pretopological stacks" in [Noo05b] and his "topological stacks" satisfy a stronger condition.

We write $\text{S}\textsc{tacks}^{\mathbf{S}}$ for the category of stacks on **S** which admit an atlas. Note that $\text{S}\textsc{tacks}^{\mathbf{top}}$ contains the category of spaces as a full subcategory. A topological (or differentiable, resp.) stack is said to be a *Deligne-Mumford* stack if it has an étale atlas, i.e. there is an atlas $p : \mathfrak{X} \to X$ which is a local homeomorphism (local diffeomorphism, resp.). A differentiable Deligne-Mumford stack is the same as an *orbifold*.

There is also the category $\text{S}\textsc{tacks}^{\mathbf{sch}}$ of algebraic stacks, studied in the book [LMB00]. Moduli stacks of (stable) curves, which constitute the example of interested to us, are most conveniently described (and constructed) as algebraic stacks. There is a functor $\text{S}\textsc{tacks}^{\mathbf{sch}} \to \text{S}\textsc{tacks}^{\mathbf{top}}$ which extends the "complex points functor" and is constructed as follows (for details, see [Noo05b], p. 78 f.). An atlas $X \to \mathfrak{X}$ gives rise to a groupoid object in schemes $X \times_{\mathfrak{X}} X \rightrightarrows X$, and the moduli stack of torsors for this groupoid object is canonically equivalent to the original stack. Taking complex points with the analytic topology gives a groupoid in topological spaces which determines a topological stack. The restriction of this functor to smooth stacks in schemes takes values in differentiable stacks, and its restriction to smooth Deligne-Mumford algebraic stacks takes values in differentiable Deligne-Mumford stacks.

2.2. **The homotopy type of a topological stack.**

We now introduce the homotopy type of a topological stack. There is a folklore definition of the homotopy type as the classifying space of the groupoid associated to an atlas. We present an axiomatic approach which is equivalent by Proposition 2.6. The content here is a ideological reemphasis of ideas which have been present in the literature for some time. The main technical points of the following exposition are contained in [Noo05a], although the notion of a homotopy type does not occur explicitly there.

Definition 2.2. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a representable morphism of topological stacks. Then f is said to be a *universal weak equivalence* if for any test map $Y \to \mathfrak{Y}$ from a space Y, the left vertical map in the diagram

$$\begin{tikzcd} Y \times_{\mathfrak{Y}} \mathfrak{X} \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d, "f"] \\ Y \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

is a weak homotopy equivalence of topological spaces.

A *homotopy type* for a topological stack $\mathfrak{X}$ is a pair $(\mathrm{Ho}(\mathfrak{X}), \eta)$, where $\mathrm{Ho}(\mathfrak{X})$ is a CW complex and $\eta : \mathrm{Ho}(\mathfrak{X}) \to \mathfrak{X}$ is a universal weak equivalence (which is automatically representable, by [Noo05b], Corollary 7.3).

Definition 2.3. Let $\mathfrak{X}$ be a topological stack and Y be a topological space. A *concordance* between elements $t_0, t_1 \in \mathfrak{X}(Y)$ is an element $t \in \mathfrak{X}(Y \times [0,1])$, together with isomorphisms $t|_{Y \times \{i\}} \cong t_i$, $i = 0, 1$. The category $\mathfrak{X}(Y)$ is skeletally small and concordance is an equivalence relation on the objects. The set of concordance classes of objects is denoted $\mathfrak{X}[Y]$.

Note that for spaces X and Y, there is a natural bijection between concordance classes $X[Y]$ and homotopy classes $[Y, X]$.

Lemma 2.4 ([Noo05a], Corollary 3.8). *Let $\eta : \mathrm{Ho}(\mathfrak{X}) \to \mathfrak{X}$ be a homotopy type for $\mathfrak{X}$. Then for each CW complex Y and map $g : Y \to \mathfrak{X}$, there exists a map $h : Y \to \mathrm{Ho}(\mathfrak{X})$ and a concordance between $\eta \circ h$ and g. Moreover, h is unique up to homotopy (which is the same as concordance).*

In particular, there is a natural bijection between the set of concordance classes $\mathfrak{X}[Y]$ and the set of homotopy classes of maps $[Y; \mathrm{Ho}(\mathfrak{X})]$ when Y is a CW complex.

Corollary 2.5. *Any two homotopy types of a topological stack are canonically homotopy equivalent. Moreover, choosing homotopy types defines a functor from the category of stacks over **top** which admit a homotopy type to the homotopy category of spaces. This functor sends 2-isomorphic morphisms of stacks to identical homotopy classes of maps.*

Proof. This corollary follows immediately from Lemma 2.4 (for the last sentence, note that 2-isomorphic morphisms are concordant). □

We have not yet seen that a topological stack admits a homotopy type; this is answered by Theorem 2.7 below.

Let $\mathfrak{X}$ be a topological stack with an atlas $X_0 \to \mathfrak{X}$. This determines a simplicial space $X_n = X_0 \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} X_0$ ($n+1$ copies) which is in fact the nerve of the topological groupoid $X_1 = X_0 \times_{\mathfrak{X}} X_0 \rightrightarrows X_0$. Let $\|X_\bullet\|$ be the *thick* realization of the simplicial space $X_\bullet$. The thick realization of a simplicial space is obtained by forgetting the degeneracies and using only the boundary maps. In most cases of interest, the thick geometric realization and the usual geometric realization are homotopy equivalent, see [Seg74, p. 308].

Proposition 2.6 ([Noo05a], Theorem 3.11). *If $X_0 \to \mathfrak{X}$ is an atlas of a topological stack with associated simplicial space $X_\bullet$, then there is a universal weak equivalence $\|X_\bullet\| \to \mathfrak{X}$.*

The space $\|X_\bullet\|$ is in general not a CW complex. To produce a homotopy type, we need a small extra argument. The realization of the singular simplicial set $|Sing_\bullet(\|X_\bullet\|)|$ is a CW complex and the evaluation map $|Sing_\bullet(\|X_\bullet\|)| \to \|X_\bullet\|$ is both a weak homotopy equivalence and a Serre fibration. Therefore, the composition $|Sing_\bullet(\|X_\bullet\|)| \to \mathfrak{X}$ is a homotopy type. This shows:

Theorem 2.7. *Any topological stack admits homotopy type.*

Homotopy types and group actions. There is a pleasant interaction between the notion of the homotopy type of a stack and more familiar topological constructions.

Firstly, if X is a CW complex then we can consider X as a topological stack. Clearly, the identity map $X \to X$ is a universal weak equivalence and thus $\mathrm{Ho}(X) \simeq X$.

An important class of examples of stacks are the *(global) quotient stacks*. Let G be a topological group acting on a space X. The quotient stack $X/\!\!/G$ is defined as follows. If Y is space, then $X/\!\!/G(Y)$ is the groupoid of triples (P, p, f); $p : P \to Y$ a principal G-bundle and $f : P \to X$ a G-equivariant map. The isomorphisms are defined in the obvious way. There is a natural morphism $q : X \to X/\!\!/G$ defined as follows: Consider

the trivial principal G-bundle $pr_X : G \times X \to X$. Note that G acts on $G \times X$ only by group multiplication (and not on X!) and that the action map $\mu : G \times X \to X$ is G-equivariant. Thus $(G \times X, pr_X, \mu)$ is an element of $X/\!\!/G(X)$, defining a morphism $q : X \to X/\!\!/G$. Note that q is a principal G-bundle.

Proposition 2.8. *The homotopy type of $X/\!\!/G$ is homotopy equivalent to the Borel construction $EG \times_G X$.*

Proof. The projection map $EG \times X \to X$ is G-equivariant while the quotient map $EG \times X \to EG \times_G X$ is a principal G-bundle, so both maps together define a morphism

$$\eta : EG \times_G X \to X/\!\!/G.$$

Clearly, η is a fiber bundle with structure group G and fiber EG: it is associated to the principal bundle $X \to X/\!\!/G$. Therefore, if Y is a space and $Y \to X/\!\!/G$ a map, then the pullback $Y \times_{X/\!\!/G} (EG \times_G X) \to Y$ is a fiber bundle with contractible fibers, hence a weak homotopy equivalence. Hence η is a universal weak equivalence. □

An important quotient stack is the moduli stack $\mathfrak{M}_{g,n}$ of smooth complex curves. It is the stack quotient of the Teichmüller space $\mathfrak{T}_{g,n}$ by the action of the mapping class group Γ_g^n of isotopy classes of orientation preserving diffeomorphism of a genus g surface with n marked points. Hence

$$\mathrm{Ho}(\mathfrak{M}_{g,n}) \simeq E\Gamma_g^n \times_{\Gamma_g^n} \mathfrak{T}_{g,n} \simeq B\Gamma_g^n,$$

because the Teichmüller space is contractible.

We will have occasion to deal with group actions on stacks. Suppose $\mathfrak{X}$ is a topological stack with a strict action of a group G (i.e. the action is not just up to coherent 2-morphisms). We will not have to care about group actions which are not strict. Given a strict G-action on $\mathfrak{X}$ and a G-space Y, the notion of an equivariant morphism $Y \to \mathfrak{X}$ is well-defined.

There are two equivalent descriptions of principal G-bundles over a stack $\mathfrak{X}$: as a morphism $\mathfrak{X} \to */\!\!/G$, or as a stack $\mathfrak{P}$ with a strict G-action and a G-invariant representable morphism $\mathfrak{P} \to \mathfrak{X}$ such that the pullback $\mathfrak{P} \times_{\mathfrak{X}} X \to X$ along any morphism $X \to \mathfrak{X}$ is a principal G-bundle in the usual sense. An analogous remark applies to arbitrary fiber bundles.

The quotient stack $\mathfrak{X}/\!\!/G$ is defined in the same way as $X/\!\!/G$ for spaces X: for a space Y, an object of $(\mathfrak{X}/\!\!/G)(Y)$ consists of a principal G-bundle $P \to Y$ and a G-equivariant morphism $P \to \mathfrak{X}$. Again, it is clear that $\mathfrak{X} \to \mathfrak{X}/\!\!/G$ is a principal G-bundle.

Proposition 2.9. *Let $\mathfrak{X}$ be a topological stack with a strict G-action. Then the following hold.*

1. *$\mathfrak{X}/\!\!/G$ is a also a topological stack.*
2. *There exists a homotopy type $\mathrm{Ho}(\mathfrak{X})$ which is a principal bundle on $\mathrm{Ho}(\mathfrak{X}/\!\!/G)$ such that the universal morphism $\mathrm{Ho}(\mathfrak{X}) \to \mathfrak{X}$ is G-equivariant.*
3. $\mathrm{Ho}(\mathfrak{X}/\!\!/G) \simeq EG \times_G \mathrm{Ho}(\mathfrak{X})$.

Proof. Let $X \to \mathfrak{X}$ be an atlas, i.e. a representable morphism which admits local sections. Because $\mathfrak{X} \to \mathfrak{X}/\!\!/G$ is a bundle, the composite $X \to \mathfrak{X}/\!\!/G$ is clearly a representable morphism with local sections. This shows (1).

For (2), choose a homotopy type $\mathrm{Ho}(\mathfrak{X}/\!\!/G) \to \mathfrak{X}/\!\!/G$ and consider the fiber-square

$$\begin{tikzcd} \mathrm{Ho}(\mathfrak{X}/\!\!/G) \times_{\mathfrak{X}/\!\!/G} \mathfrak{X} \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] \\ \mathrm{Ho}(\mathfrak{X}/\!\!/G) \arrow[r] & \mathfrak{X}/\!\!/G. \end{tikzcd}$$

Because the right vertical map is a principal G-bundle, so is the left vertical map. Because the bottom horizontal map is a universal weak equivalence, the top horizontal is also a universal weak equivalence. Thus the space $\mathrm{Ho}(\mathfrak{X}/\!\!/G) \times_{\mathfrak{X}/\!\!/G} \mathfrak{X} \to \mathfrak{X}$ is a homotopy type for $\mathfrak{X}$ and is also G-equivariant, which shows (2).

For (3), observe that the natural map $EG \times_G \mathrm{Ho}(\mathfrak{X}) \to \mathrm{Ho}(\mathfrak{X})/G = \mathrm{Ho}(\mathfrak{X}/\!\!/G)$ is a fiber bundle with fiber EG, hence a weak homotopy equivalence. □

Homology of a topological stack. The definition of the homotopy type of a stack is justified both by the above examples and by the fact, which we now explain, that the space $\mathrm{Ho}(\mathfrak{X})$ has the correct (co)homology. A topological stack has singular (co)homology and sheaf cohomology. These turn out to be canonically isomorphic to the (co)homology of the space $\mathrm{Ho}(\mathfrak{X})$.

The following definition of singular homology for stacks is from [Beh04]. An atlas $X \to \mathfrak{X}$ determines a simplicial space $X_\bullet$. Applying $\mathrm{Sing}_\bullet$ produces a bisimplicial set which generates a double complex $C_{\bullet,\bullet}(\mathfrak{X})$ of Abelian groups. The singular homology $H_*^{sing}(\mathfrak{X})$ of $\mathfrak{X}$ is defined to be the homology of the total complex $Tot(C_{\bullet,\bullet}(\mathfrak{X}))$. It can be shown that this is independent of the choice of atlas. There is a map of simplicial spaces

$$(p \mapsto |\,\mathrm{Sing}_\bullet X_p|) \to (p \mapsto X_p);$$

the homology of the (realization of the) left hand side is $H_*^{sing}(\mathfrak{X})$, the homology of the right-hand side is $H_*(\mathrm{Ho}(\mathfrak{X}))$. A straightforward application of the homology spectral sequence of a simplicial space [Seg68] shows that the induced map on homology is an isomorphism. Thus we have a natural isomorphism

$$H_*^{sing}(\mathfrak{X}) = H_*(Tot(C_{\bullet,\bullet}(\mathfrak{X}))) \cong H_*(|X_\bullet|) = H_*(\mathrm{Ho}(\mathfrak{X})). \tag{2.10}$$

The singular cohomology of $\mathfrak{X}$ is defined analogously and it agrees with the sheaf cohomology by standard arguments. By the same reasoning as before, the singular cohomology is canonically isomorphic to $H^*(\mathrm{Ho}(\mathfrak{X}))$.

For a topological stack $\mathfrak{X}$, let $\mathfrak{X}^{coarse}$ be the coarse moduli space (this is the orbit space of a groupoid presenting $\mathfrak{X}$). There is a natural map $\mathfrak{X} \to \mathfrak{X}^{coarse}$ (which is almost never representable) and the composition

$$\mu_{\mathfrak{X}} : \mathrm{Ho}(\mathfrak{X}) \to \mathfrak{X} \to \mathfrak{X}^{coarse} \tag{2.11}$$

is a rational homology equivalence when $\mathfrak{X}$ is an orbifold (see e.g. [Hae84]).

When $\mathfrak{X}$ is an orbifold it has an orbifold fundamental group $\pi_1^{orb}\mathfrak{X}$ (see [Moe02]), and there is a natural isomorphism $\pi_1 \mathrm{Ho}(\mathfrak{X}) \cong \pi_1^{orb}\mathfrak{X}$. One can introduce coefficient systems, and the isomorphisms (2.10) of (co)homology hold also for twisted coefficients.

3. THE PONTRJAGIN-THOM CONSTRUCTION FOR DIFFERENTIABLE STACKS

In this section we describe an extension of the classical Pontrjagin-Thom construction of homotopy-theoretic wrong-way maps to the setting of differentiable stacks.

3.1. **Preliminaries on stable vector bundles and Thom spectra.** If $W \to X$ is a real vector bundle then the *Thom space* of W, denoted X^W is the space obtained by taking the fiberwise one-point compactification of W and then collapsing the section at infinity to the basepoint. (If X is compact then this is simply the one-point compactification of W.)

A *virtual vector bundle* on a space X is a pair (E_0, E_1) of real vector bundles on X; one should think of it as the formal difference $E_0 - E_1$, and we will sometimes use this more suggestive notation. The *rank* of (E_0, E_1) is the difference $\dim E_0 - \dim E_1$. An isomorphism $(E_0, E_1) \to (F_0, F_1)$ is represented by a pair (V, θ) where V is a vector bundle and

$$\theta : E_0 \oplus F_1 \oplus V \to E_1 \oplus F_0 \oplus V$$

is a bundle isomorphism. Two pairs (θ, V), (θ', V') represent the same morphism if there exists a vector bundle U such that $V' = V \oplus U$ and $\theta' = \theta \oplus \mathrm{id}_U$ (and then take the equivalence relation that this generates). The composition of $\theta : E_0 \oplus F_1 \oplus V \to E_1 \oplus F_0 \oplus V$ and $\phi : F_0 \oplus G_1 \oplus W \to F_1 \oplus G_0 \oplus W$ is defined to be $F_1 \oplus V \oplus W$ together the composition

$$\begin{aligned} E_0 \oplus F_1 \oplus G_1 \oplus V \oplus W &\xrightarrow{\theta \oplus \mathrm{id}_{G_1 \oplus W}} E_1 \oplus F_0 \oplus G_1 \oplus V \oplus W \\ &\xrightarrow{\phi \oplus \mathrm{id}_{E_1 \oplus V}} E_1 \oplus F_1 \oplus G_0 \oplus V \oplus W. \end{aligned}$$

The category of virtual vector bundles over a fixed space is a groupoid; these form a presheaf of groupoids on the site **top**. Let $\mathfrak{K}$ denote the stackification of the above presheaf. The objects of this stack are slightly more general than virtual bundles; they can locally be presented as formal differences of vector bundles, but globally this might be impossible. Objects of $\mathfrak{K}$ are called *stable vector bundles*.

Let $\mathfrak{K}_d$ denote the full substack consisting of virtual bundles of rank d. For $n \geq d$, let $* \to \mathfrak{K}_d$ be the arrow representing the stable vector bundle $(\mathbb{R}^n; \mathbb{R}^{n-d})$. It is easy to see that this is an atlas for $\mathfrak{K}_d$ (as a topological stack) and in fact $\mathfrak{K}_d$ is equivalent to the stack $*/\!/O$. Thus

$$\mathrm{Ho}(\mathfrak{K}) = \coprod_{d \in \mathbb{Z}} \mathrm{Ho}(\mathfrak{K}_d) \simeq \mathbb{Z} \times BO,$$

as expected, and 2-isomorphism classes of morphisms $X \to \mathfrak{K}$ correspond to homotopy classes $X \to \mathbb{Z} \times BO$.

For any map $c_W : X \to \{d\} \times BO$ which classifies a stable vector bundle W of rank d, there is an associated *Thom spectrum* $\mathbb{T}\mathbf{h}(W)$, produced as follows. There is an exhaustive filtration $X_{-d} \subset X_{1-d} \subset \cdots \subset X$, where $X_n := c_W^{-1}(\{d\} \times BO_{d+n})$. Let $W_n := c_W^* \gamma_{d+n}$ be the pullback of the $d+n$-dimensional universal vector bundle. Clearly, there is an isomorphism $W_{n+1}|_{X_n} \cong \mathbb{R} \oplus W_n$. The n^{th} space of $\mathbb{T}\mathbf{h}(W)$ is $X_n^{W_n}$ and the structure maps are

$$\Sigma X_n^{W_n} \cong X_n^{\mathbb{R} \oplus W_n} \cong X_n^{W_{n+1}|_{X_n}} \hookrightarrow X_{n+1}^{W_{n+1}}.$$

The homotopy type of the spectrum $\mathbb{T}\mathbf{h}(W)$ depends only on the homotopy class of c_W, which can be viewed as an element in the real K-theory group $KO^0(X)$. Furthermore,

when W is representable by an actual vector bundle W_0 then the Thom spectrum is homotopy equivalent to the suspension spectrum $\Sigma^\infty X^{W_0}$ of the Thom space. The reader who wants to know more details about Thom spectra is advised to consult [Rud98], chapter IV, §5.

3.2. **The classical Pontrjagin-Thom construction.** We briefly recall the classical construction. Let $f : M \to N$ be a proper smooth map between smooth manifolds of codimension d (i.e., $\dim N - \dim M = d$). The normal bundle

$$\nu(f) := f^*TN - TM$$

is a virtual vector bundle of dimension d on M. For n large enough there exists an embedding $j : M \hookrightarrow \mathbb{R}^n \times N$ such that $pr_N \circ j = f$. The virtual bundle $\nu(j) - \mathbb{R}^n$ is canonically isomorphic to $\nu(f)$.

Choose a tubular neighborhood U of $j(M)$, identify $U \cong \nu(j)$, and define a map $\mathbb{R}^n \times N \to M^{\nu(j)}$ as follows: if a point lies in U then it is mapped to the corresponding point in $\nu(j) \subset M^{\nu(j)}$; all points outside U are mapped to the basepoint of $M^{\nu(j)}$. Because f is proper, this construction extends to a map $\Sigma^n N_+ \to M^{\nu(j)}$. The space $M^{\nu(j)}$ is the n^{th} space of the spectrum $\mathbb{T}\mathbf{h}(\nu(f))$ and letting n tend to infinity defines a map of spectra

$$\mathrm{PT}_f : \Sigma^\infty N_+ \to \mathbb{T}\mathbf{h}(\nu(f))$$

which is the classical Pontrjagin-Thom map. Recall that the functor Σ^∞ from spaces to spectra is left adjoint to the functor Ω^∞. The adjoint map of PT_f is a map of spaces $N \to \Omega^\infty \mathbb{T}\mathbf{h}(\nu(f))$, which we also denote by PT_f, because there is no risk of confusion. The homotopy class of PT_f does not depend on the choices involved. The Pontrjagin-Thom map can be used to define umkehr maps in cohomology, see section 6.

3.3. **Normal bundles for stacks and statement of the theorem.** To extend the Pontrjagin-Thom construction to stacks one must be able to define the normal bundle of a morphism.

Let $f : \mathfrak{X} \to \mathfrak{Y}$ be proper representable morphism of differentiable stacks. The *codimension* d of f is by definition $d = \dim(Y) - \dim(Y \times_{\mathfrak{Y}} \mathfrak{X})$, where $Y \to \mathfrak{Y}$ is an atlas. Let $Y \to \mathfrak{Y}$ be an atlas and let $X := \mathfrak{X} \times_{\mathfrak{Y}} Y \to \mathfrak{X}$ be the induced atlas for $\mathfrak{X}$. The map f pulls back to a map $f_Y : X \to Y$ which is a proper smooth map. The normal bundle $f^*TY - TX$ is a virtual vector bundle on X, and so it is classified by a morphism $X \to \mathfrak{K}_d$. Since normal bundles are natural with respect to pullback along submersions, this morphism descends to a morphism

$$\nu(f) : \mathfrak{X} \to \mathfrak{K}_d$$

Taking homotopy types produces a map $\mathrm{Ho}(\mathfrak{X}) \to BO$ which then yields a Thom spectrum $\mathbb{T}\mathbf{h}(\nu(f))$.

If N is a manifold and $g : N \to \mathfrak{Y}$ is a map which is transversal to f. Then we have a pullback diagram

$$\begin{tikzcd} N \times_{\mathfrak{Y}} \mathfrak{X} \arrow[r, "h"] \arrow[d, "f_N"] & \mathfrak{X} \arrow[d, "f"] \\ N \arrow[r, "g"] & \mathfrak{Y}, \end{tikzcd}$$

where f_N is a proper map of manifolds. Thus we have a Pontrjagin-Thom map $\mathrm{PT}_{f_N} : N \to \Omega^\infty \mathbb{T}\mathbf{h}(\nu(f_N))$. There is an induced morphism $\nu(f_N) \to \nu(f)$ of stable vector

bundles which covers the map h and induces a map $\Omega^\infty\mathbb{T}\mathbf{h}(\nu(f_N)) \to \Omega^\infty\mathbb{T}\mathbf{h}(\nu(f))$. Finally recall that g has a canonical (up to homotopy) lift $g' : Y \to \mathrm{Ho}(\mathfrak{Y})$.

Definition 3.1. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a proper representable morphism of differentiable stacks. A *Pontrjagin-Thom map* for f is a map $\mathrm{PT}_f : \mathrm{Ho}(\mathfrak{Y}) \to \Omega^\infty\mathbb{T}\mathbf{h}(\nu(f))$ such that the following diagram is homotopy-commutative:

$$\begin{tikzcd}
\Omega^\infty\mathbb{T}\mathbf{h}(\nu(f_N)) \arrow[r] & \Omega^\infty\mathbb{T}\mathbf{h}(\nu(f)) \\
N \arrow[u, "\mathrm{PT}_{f_N}"] \arrow[r, "g'"] & \mathrm{Ho}(\mathfrak{Y}) \arrow[u, "\mathrm{PT}_f"']
\end{tikzcd}$$

where g' is a lift of a map $g : N \to \mathfrak{Y}$ that is transveral to f.

The following is the main result of this section.

Theorem 3.2. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a proper representable morphism of differentiable stacks with $\mathfrak{Y}$ a local quotient stack (see Definition 3.3 below). Then there exists a Pontrjagin-Thom map*

$$\mathrm{PT}_f : \mathrm{Ho}(\mathfrak{Y}) \to \Omega^\infty\mathbb{T}\mathbf{h}(\nu(f)).$$

The map PT_f is unique in the sense that it depends on a contractible space of choices. The main ingredient in the proof of this theorem is a variant of the Whitney Embedding Theorem for local quotient stacks (Prop 3.11). One constructs appropriate embeddings for global quotients using standard equivariant techniques and then glues these together to obtain embeddings for local quotient stacks. The construction of Pontrjagin-Thom maps is then a matter of adapting the classical construction.

3.4. **Local quotient stacks.**

Here we introduce local quotient stacks, which we view as the natural setting for the Pontrjagin-Thom construction.

Definition 3.3. A *local quotient stack* is a topological stack $\mathfrak{X}$, such that

(1) there exists a paracompact atlas for $\mathfrak{X}$.
(2) there exists a countable cover of open substacks $\mathfrak{X}_i \subset \mathfrak{X}$ such that $\mathfrak{X}_i \cong X_i /\!\!/ G_i$ for some space X_i and some compact Lie group G_i.
(3) The diagonal morphism $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representable and proper.

A differentiable stack is a local quotient stack if the spaces X_i are smooth manifolds with smooth G_i-actions.

Lemma 3.4 ([FHT], Lemma A.14)**.** *If $\mathfrak{Y}$ is a local quotient stack and $f : \mathfrak{X} \to \mathfrak{Y}$ is a representable morphism of topological stacks then $\mathfrak{X}$ is a local quotient stack as well. In particular, every open substack of a local quotient stack is a local quotient stack. The analogous statements for differentiable local quotient stacks are also true.*

Proof. First suppose that $\mathfrak{Y}$ is a global quotient $Y/\!\!/G$. Let $X := \mathfrak{X} \times_{\mathfrak{Y}} Y \to \mathfrak{X}$ be the atlas of $\mathfrak{X}$ obtained by pulling back the atlas $Y \to \mathfrak{Y}$. One easily checks that $X \times_{\mathfrak{X}} X \cong G \times X$ and that the two arrows $X \times_{\mathfrak{X}} X = G \times X \rightrightarrows X$ are the projection onto X and a group action. Furthermore, one can check that $f : \mathfrak{X} \to \mathfrak{Y}$ is represented by a G-equivariant map $X \to Y$. Now suppose $\mathfrak{Y}$ is a local quotient stack with a covering by global quotients $\{\mathfrak{Y}_i \cong Y/\!\!/G_i\}$. The substacks $\mathfrak{X}_i := \mathfrak{Y}_i \times_{\mathfrak{Y}} \mathfrak{X}$ form an open cover of $\mathfrak{X}$ and by the above, $\mathfrak{X}_i \cong X_i/\!\!/G_i$. $\square$

Lemma 3.4 indicates that the class of local quotient stack is large and robust. Orbifolds are local quotient stacks and so are global quotient stacks of the form $Y /\!\!/ \Gamma$, where Γ is a (possibly noncompact) Lie group which acts properly on Y. A very general result by Zung [Zun06] states that any proper Lie groupoid represents a local quotient stack.

Lemma 3.5. *The coarse moduli space $\mathfrak{X}^{coarse}$ of a differentiable local quotient stack $\mathfrak{X}$ is a paracompact Hausdorff space.*

Proof. Given an atlas $X \to \mathfrak{X}$, one sees that the associated groupoid $X \times_{\mathfrak{X}} X \rightrightarrows X$ is proper in the sense of [Moe02]. The coarse moduli space is the orbit space of this groupoid and hence it is Hausdorff and paracompact. □

As an application of this lemma, we have existence of locally finite smooth partitions of unity subordinate to any open cover of $\mathfrak{Y}$ as follows. Any open cover of $\mathfrak{Y}$ gives an open cover of $\mathfrak{Y}^{coarse}$. On $\mathfrak{Y}^{coarse}$, we have partitions of unity, which can then be pulled back via $\mathfrak{Y} \to \mathfrak{Y}^{coarse}$.

3.5. **Universal vector bundles on local quotient stacks.** In this section, we introduce "universal vector bundles" on stacks. Freed, Hopkins and Teleman [FHT] showed that any local quotient stack admits a universal *Hilbert* bundle. For the purpose of constructing Pontrjagin-Thom maps we instead need universal vector bundles with fiber $\mathbb{R}^\infty = \operatorname{colim} \mathbb{R}^n$. Here we show that any local quotient stack has a universal countably-dimensional vector bundle (its completion will be a universal Hilbert bundle).

Consider $\mathbb{R}^\infty$ with the colimit topology; this is a complete, locally convex topology which is not metrizable. It is a very fine topology: *any* linear map $\mathbb{R}^\infty \to V$ to an arbitrary topological vector space V is continuous.

Consider the group $O(\mathbb{R}^\infty)$ of isometries of $\mathbb{R}^\infty$ with respect to the standard inner product. On $O(\mathbb{R}^\infty)$ we define the following topology. The compact-open topology on the vector space $\operatorname{End}(\mathbb{R}^\infty)$ agrees with the topology of pointwise convergence. Embed $O(\mathbb{R}^\infty) \hookrightarrow \operatorname{End}(\mathbb{R}^\infty) \times \operatorname{End}(\mathbb{R}^\infty)$ via $f \mapsto (f, f^{-1})$ and take the induced subspace topology on $O(\mathbb{R}^\infty)$. Finally, replace this topology by its compactly generated replacement [Ste67].

Proposition 3.6. *Let $O(\mathbb{R}^\infty)$ be endowed with the topology described above.*

(1) *$O(\mathbb{R}^\infty)$ is a topological group.*
(2) *By extension, $O(\mathbb{R}^\infty)$ acts by isometries on ℓ^2 and this action is continuous*[1].
(3) *Let $V \subset \mathbb{R}^\infty$ be a finite-dimensional subspace. Then the standard inclusion $O(V) \to O(\mathbb{R}^\infty)$ is continuous.*
(4) *Let G be a compact Lie group and let V be a countably-dimensional orthogonal G-representation (then V is isometric to $\mathbb{R}^\infty$). The action homomorphism $G \to O(V)$ is continuous.*

Proof. Claim (1) follows immediately from Theorem 5.9 in [Ste67]. For (2), by Theorem 3.2.(i) of [Ste67], it suffices to show that $O(\mathbb{R}^\infty) \times \ell^2 \to \ell^2$ is continuous when $O(\mathbb{R}^\infty)$ has the compact-open topology, which is straightforward. Claim (3) is immediate, as

[1] This action cannot be extended continuously to an action of the group of all (say bounded) isomorphisms of $\mathbb{R}^\infty$.

is (4) because G is compact and any finite-dimensional representation is continuous by definition. $\square$

By *countably-dimensional vector bundle* we shall always mean a fiber bundle with structure group $O(\mathbb{R}^\infty)$ (with the above topology) and fiber $\mathbb{R}^\infty$.

Definition 3.7. Let $\mathfrak{X}$ be a topological stack. A *universal vector bundle* on $\mathfrak{X}$ is a countably-dimensional vector bundle $\mathfrak{H} \to \mathfrak{X}$ such that any other (finite or countably dimensional) vector bundle $\mathfrak{E} \to \mathfrak{X}$ admits a complemented isometric embedding $\mathfrak{E} \hookrightarrow \mathfrak{H}$.

A universal vector bundle $\mathfrak{H}$ has two "absorption properties": the tensor product $\mathfrak{H} \otimes \mathbb{R}^\infty$ is isometrically isomorphic to $\mathfrak{H}$ and so is the sum $\mathfrak{H} \oplus \mathfrak{E}$ with any countably-dimensional vector bundle $\mathfrak{E}$.

Proposition 3.8. *If $\mathfrak{X}$ is a local quotient stack, then there exists a universal vector bundle $\mathfrak{H} \to \mathfrak{X}$. Furthermore, it is unique up to isometry. For any other vector bundle $\mathfrak{E} \to \mathfrak{X}$, the space of isometric embeddings $\mathrm{Mon}(\mathfrak{E}; \mathfrak{H})$ is weakly contractible. The pullback along any representable morphism of local quotient stacks is also a universal vector bundle.*

Remark 3.9. The local quotient hypothesis is necessary. An example of a differentiable stack which is not a local quotient and does not admit a universal vector bundle is $*/\!\!/\mathbb{R}$ because the set of equivalence classes of orthogonal $\mathbb{R}$-representations is uncountable.

If countably-dimensional vector bundles are replaced by (separable) Hilbert space bundles, then the analogue of Proposition 3.8 is proven in [FHT], p. 57 ff. Due to Lemma 3.6, (2), any countably-dimensional vector bundle can be completed to a Hilbert space bundle. When one completes the universal countably-dimensional bundle, one obtains the universal Hilbert bundle. Hence Proposition 3.8 can be interpreted as the statement that the universal Hilbert-bundle on a local quotient stack admits a "countable substructure".

To prove Proposition 3.8, the arguments of loc. cit. carry over to our situation without essential change, with one exception: the construction of a universal vector bundle on $X/\!\!/G$ needs to be adjusted.

Let G be a compact Lie group, let $L^2(G)$ be the regular representation of G and $L^2(G)_{fin} \subset L^2(G)$ be the space of G-finite vectors. It is an L^2-dense, countably-dimensional subspace (see [BtD95], Theorem 5.7). Moreover, it contains any irreducible G-representation with finite multiplicity. Thus $L^2(G)_{fin} \otimes \mathbb{R}^\infty$ contains any countably-dimensional G-representation.

Lemma 3.10. *If X is a paracompact space and G is a compact Lie group acting continuously on X, then the vector bundle $X \times_G (L^2(G)_{fin} \otimes \mathbb{R}^\infty) \to X/\!\!/G$ is a universal vector bundle.*

Proof. This is only a slight modification of well-known results. First, let G be the trivial group and let $\pi : V \to X$ be a vector bundle. Choose a countable cover $\{U_i\}_{i\in\mathbb{N}}$ of X, trivializations $(\pi, \phi_i) : V|_{U_i} \to U_i \times \mathbb{R}^\infty$, and a locally finite partition of unity (λ_i) subordinate to the chosen cover. Then the map $\phi : V \to X \times \mathbb{R}^{\infty\times\infty}$ given by

$$\phi(v) := \Big(\pi(v), \sqrt{\lambda_1(\pi(v))}\phi_1(v), \sqrt{\lambda_2(\pi(v))}\phi_2(v), \sqrt{\lambda_3(\pi(v))}\phi_3(v), \dots\Big)$$

is clearly an isometric complemented embedding.

For nontrivial G, we argue as follows. Let $V \to X$ be a G-equivariant vector bundle and let $j : V \hookrightarrow X \times \mathbb{R}^\infty$ be an embedding as constructed above. By Frobenius reciprocity [BtD95], p. 144, there is a continuous isomorphism $\mathrm{Hom}(W, U) \cong \mathrm{Hom}_G(W; U \otimes L^2(G)_{fin})$ for all countably-dimensional G-modules W and all countably-dimensional vector spaces U. Use this to extend j to a G-equivariant embedding $j' : V \hookrightarrow X \times \mathbb{R}^\infty \otimes L^2(G)_{fin}$. □

Proposition 3.8 now follows from Lemma 3.10 by the same arguments as used in [FHT].

3.6. The Whitney embedding theorem for local quotient stacks.

Proposition 3.11. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a proper representable morphism between differentiable local quotient stacks. Let $\pi : \mathfrak{H} \to \mathfrak{Y}$ be a universal vector bundle. Then there exists a fat embedding $\mathfrak{X} \hookrightarrow \mathfrak{H}$ over $\mathfrak{Y}$. More precisely, there exists a countably-dimensional vector bundle $q : \mathfrak{E} \to \mathfrak{X}$ with zero section s and an open embedding $j : \mathfrak{E} \hookrightarrow \mathfrak{H}$ such that $\pi \circ j \circ s = f$. Moreover, the space of such embeddings is contractible.*

Proof. First assume that $\mathfrak{Y} = Y /\!/ G$, where G is a compact Lie group acting on Y with finite orbit type, i.e. the number of conjugacy classes of isotropy subgroups is finite. Then $\mathfrak{X} \cong X /\!/ G$ and f is represented by a G-equivariant map $X \to Y$ (compare proof of Lemma 3.4). Because f is proper, the action on X also has finite orbit type. Mostow showed ([Mos57], p. 444 f) that there exists a finite-dimensional G-representation V and a G-equivariant embedding $X \hookrightarrow Y \times V$. The vector bundle $Y \times_G V$ admits an isometric embedding into the universal bundle $Y \times_G L^2(G)_{fin} \otimes \mathbb{R}^\infty$. There exist G-equivariant tubular neighborhoods; the choice of one gives a fat embedding. The space of all G-equivariant tubular neighborhoods is contractible. A variant of the Eilenberg swindle (compare the proof Lemma A.35 in [FHT]) shows that the space of all fat embeddings $\mathfrak{X} \to \mathfrak{H}$ over $\mathfrak{Y}$ is contractible.

If $\mathfrak{Y}$ is a local quotient stack then we can glue these locally defined fat embeddings by the following procedure. Choose open substacks $W_i /\!/ G_i = \mathfrak{W}_i \subset \mathfrak{U}_i = U_i /\!/ G_i$, $i \in \mathbb{N}$ of $\mathfrak{Y}$ such that $W_i \subset U_i$ is relatively compact and such that the collection of all $\mathfrak{W}_i$ covers $\mathfrak{Y}$. Then the G_i-action on W_i is of finite orbit type and therefore the space of fat embeddings $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{W}_i \to \mathfrak{H}|_{\mathfrak{W}_i}$ is contractible (in particular, nonempty).

For any finite nonempty $S \subset \mathbb{N}$ let $\mathfrak{W}_S$ be the intersection (alias fibered product over $\mathfrak{X}$) of all $\mathfrak{W}_i$, $i \in S$. Being an open substack of some $\mathfrak{W}_i$, $\mathfrak{W}_S$ is also a global quotient stack. We have seen that the space $\mathcal{F}_S$ of all fat embeddings $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{W}_S \hookrightarrow \mathfrak{H}|_{\mathfrak{W}_S}$ over $\mathfrak{W}_S$ is contractible for all finite nonempty subsets $S \subset \mathbb{N}$. Let $\Delta_S \subset \mathbb{R}^S$ denote the n-simplex spanned by S. Because the spaces $\mathcal{F}_S$ are all contractible, by induction on $|S|$ it is possible to choose maps $h_S : \Delta_S \to \mathcal{F}_S$ satisfying the compatibility conditions that whenever $T \subset S$ then $h_S|_{\Delta_T} = r_{T,S} \circ h_T$, where $r_{T,S} : \mathcal{F}_T \to \mathcal{F}_S$ is the obvious restriction map.

Now let $\{\lambda_i\}$ $(i \in \mathbb{N})$ be a locally finite partition of unity subordinate to the covering $\{\mathfrak{W}_i\}$. For each $x \in \mathfrak{X}$ let

$$S(x) := \{i \in \mathbb{N} \mid f(x) \in \mathrm{supp}(\lambda_i)\}.$$

For $x \in \mathfrak{X}$, we put

$$j(x) := h_{S(x)}\left(\sum_{i \in S(x)} \lambda_i(f(x))\{i\}\right)(f(x)) \in \mathfrak{H}.$$

Using that $\{\lambda_i\}_{i\in\mathbb{N}}$ is locally finite and the compatibility of the maps $\{h_S\}$, one observes that j is continuous and a fat embedding. The contractibility follows from a similar argument. □

3.7. **Constructing the Pontrjagin-Thom map.** Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a proper representable morphism of local quotient stacks, let $\mathfrak{H} \to \mathfrak{Y}$ be a universal vector bundle, $\mathfrak{E} \to \mathfrak{X}$ a vector bundle and $j : \mathfrak{E} \to \mathfrak{H}$ be a fat embedding over $\mathfrak{Y}$, as provided by Proposition 3.11. Let $Y \to \mathfrak{Y}$ be an atlas and $X \to \mathfrak{X}$ the induced atlas for $\mathfrak{X}$. Similarly, we have atlases $E \to \mathfrak{E}$ and $H \to \mathfrak{H}$. When we use these atlases to present the homotopy types, we get a diagram

$$\begin{tikzcd} \mathrm{Ho}(\mathfrak{E}) \arrow[r, "\mathrm{Ho}(j)"] \arrow[d, shift left] & \mathrm{Ho}(\mathfrak{H}) \arrow[d] \\ \mathrm{Ho}(\mathfrak{X}) \arrow[u, bend left, "s"] \arrow[r, "\mathrm{Ho}(f)"] & \mathrm{Ho}(\mathfrak{Y}), \end{tikzcd}$$

where the vertical downward maps are countably-dimensional vector bundles, the left vertical upward map is the zero section, $\mathrm{Ho}(j)$ is an open embedding and $\mathrm{Ho}(f)$ is a proper map. This square commutes (for either choice of the left vertical arrow).

The bundle $\mathrm{Ho}(\mathfrak{H})$ is the pullback of $\mathfrak{H}$ along the map $\mathrm{Ho}(\mathfrak{Y})$, therefore it is a universal vector bundle on $\mathrm{Ho}(\mathfrak{Y})$; hence $\mathrm{Ho}(\mathfrak{H}) \cong \mathrm{Ho}(\mathfrak{Y}) \times \mathbb{R}^\infty$ (this is of course not true before taking homotopy types). The proofs of Propositions 3.11 and 3.8 show that any $y \in \mathrm{Ho}(\mathfrak{Y})$ has a neighborhood U such that $\mathrm{Ho}(j) \circ s$ embeds $\mathrm{Ho}(f)^{-1}(U)$ into a finite-dimensional subbundle of $\mathrm{Ho}(\mathfrak{Y}) \times \mathbb{R}^\infty$. After passing to a smaller neighborhood and moving by an isotopy of embeddings, we can assume that

$$\mathrm{Ho}(j) \circ s : \mathrm{Ho}(f)^{-1}(U) \hookrightarrow U \times \mathbb{R}^{n_0} \subset \mathrm{Ho}(\mathfrak{Y}) \times \mathbb{R}^\infty \tag{3.12}$$

for some n_0 large enough. If $n \geq n_0$, then each space $V_{U,n} := \mathrm{Ho}(j)^{-1}(U \times \mathbb{R}^n)$ is the total space of a vector bundle on $\mathrm{Ho}(f)^{-1}(U)$. The rank of $V_{U,n}$ is equal to $d+n$, where d is the codimension of f. Clearly $V_{U,n+1} \cong V_{U,n} \oplus \mathbb{R}$. Also $V_{U,n} = V_{U',n}$ on the intersection $U \cap U'$. Therefore, the $V_{U,n}$ assemble to a stable vector bundle V on $\mathrm{Ho}(\mathfrak{X})$.

Lemma 3.13. *The stable vector bundle V is the stable normal bundle $\nu(f)$ (pulled back to $\mathrm{Ho}(\mathfrak{X})$).*

Proof. Let N be a manifold, $N \to \mathfrak{Y}$ a map transverse to f and let $N' \subset N$ be a relatively compact open submanifold. Let $f_N : M = N \times_{\mathfrak{Y}} \mathfrak{X} \to N$ be the induced map and similarly for N'. Choose a lift $N \to \mathrm{Ho}(\mathfrak{Y})$ of $N \to \mathfrak{Y}$ by the universal property of the homotopy type. The fibered product $\mathrm{Ho}(\mathfrak{Y}) \times_{\mathfrak{Y}} \mathfrak{X}$ is a model for $\mathrm{Ho}(\mathfrak{X})$ and it is easy to see that

$$\begin{tikzcd} M \arrow[r, "h"] \arrow[d, "f_N"] & \mathrm{Ho}(\mathfrak{X}) \arrow[d] \\ N \arrow[r] & \mathrm{Ho}(\mathfrak{Y}) \end{tikzcd}$$

is a pullback diagram. Clearly, the relatively compact N' maps into some $U \subset \mathrm{Ho}(\mathfrak{Y})$ as in 3.12.

It is clear that the pullback $h^* V_{U,n}$ is (canonically) isomorphic to $\nu(f_{N'}) \oplus \mathbb{R}^n$. This is precisely the same statement as saying that V is isomorphic to the homotopical realization of the stable normal bundle of f. □

The collapse construction finally defines a map of spectra $\Sigma^\infty U_+ \to \mathbb{T}\mathbf{h}(V)$ and by adjunction a map of spaces $U \to \Omega^\infty \mathbb{T}\mathbf{h}(V)$. By construction, these maps agree on intersections $U \cap U'$. This finishes the proof of Theorem 3.2.

4. The moduli stack of stable curves and graphs

The stack $\overline{\mathfrak{M}}_{g,n}$ was first constructed in the algebraic category by Deligne, Mumford and Knudsen (in [DM69] when $n = 0$ and [Knu83] for general n). We will need only the associated orbifold in the category of differentiable stacks. For more information about $\overline{\mathfrak{M}}_{g,n}$, we refer to the textbook [HM98] or the article [Edi00].

4.1. **The moduli stack of stable curves.** A nodal curve is a complete complex algebraic curve C all of whose singularities are nodal, i.e. ordinary double points. The open subset of smooth points of C will be denoted by C^{sm}. The *arithmetic genus* of a connected nodal curve is the dimension of the vector space $H^1(C, \mathcal{O}_C)$. Let $C_1, \ldots, C_k$ be the components of C, let g_i be the genus of C_i and let r be the number of nodes of C. Then the arithmetic genus is given by

$$g = \sum_{i=1}^{k} (g_i - 1) + r + 1. \tag{4.1}$$

All nodal curves in this paper are understood to be connected. Given a finite set P, a P-pointed nodal curve is a nodal curve C with an embedding of P into C^{sm}. Such a curve is *stable* if its automorphism group is finite. This means that the Euler characteristic of each component of $C^{sm} \setminus P$ is negative, or equivalently, C does not contain an irreducible component which is a projective line with fewer than 3 marked points and nodes or an elliptic curve with no marked points or nodes.

The stack $\overline{\mathfrak{M}}_{g,P}$ is the lax sheaf of groupoids on the site of schemes over $\mathbb{C}$ in the étale topology which is given by:

(1) The objects of $\overline{\mathfrak{M}}_{g,n}(X)$ are pairs $(E \xrightarrow{\pi} X, j : X \times P \hookrightarrow E)$, where π a proper morphism all of whose geometric fibers are reduced connected nodal curves of genus g, and j is an embedding over X, and each fiber is a P-pointed stable nodal curve. Such a triple a *family of pointed curves over* X.
(2) An isomorphism of families of pointed stable curves is an isomorphism of schemes over X which respects the embedding j.

Deligne-Mumford-Knudsen [DM69], [Knu83] constructed a smooth étale atlas for $\overline{\mathfrak{M}}_{g,P}$ in the category of schemes over $\operatorname{spec} \mathbb{C}$. In the complex analytic category an orbifold atlas is given by the degeneration spaces of Bers [Ber81], and another was constructed in [RS06]. The complex dimension of $\overline{\mathfrak{M}}_{g,P}$ is $3g - 3 + |P|$. An important property of this stack is that its coarse moduli space is compact.

The symmetric group Σ_P acts on $\overline{\mathfrak{M}}_{g,P}$ by permuting the marked points; thus Σ_P acts on $\mathrm{Ho}(\overline{\mathfrak{M}}_{g,P})$.

4.2. **Stable graphs.** Following [GK98], we introduce *stable graphs* as a combinatorial tool for working with the stratification of $\overline{\mathfrak{M}}_{g,n}$ and keeping track of iterations of gluing morphisms.

A *graph* Γ consists of a finite set $\mathrm{Vert}(\Gamma)$ of vertices, a finite set $H(\Gamma)$ of half-edges, together with an involution σ on $H(\Gamma)$ and a map $\tau : H(\Gamma) \to \mathrm{Vert}(\Gamma)$. The set of half-edges incident at a vertex v is $\tau^{-1}(v)$. An *edge* is a free orbit of σ and the endpoints of an edge are the vertices that its half-edges are incident at. We write $E(\Gamma)$ for the set of edges. The fixed points of σ are the *legs* of Γ.

A *stable graph* is a graph Γ, together with a function $g : \mathrm{Vert}(\Gamma) \to \mathbb{N}_{\geq 0}$ satisfying $g(v) > 0$ if the valence v is less than 3 and $g(v) > 1$ if the valence is 0. The *genus* of a connected stable graph is defined to be

$$g(\Gamma) = \sum_{v \in \mathrm{Vert}(\Gamma)} (g(v) - 1) + |E(\Gamma)| + 1.$$

We will need stable graphs equipped with an additional piece of data: a subset U of the univalent vertices of Γ and for each $u \in U$ a point $[F_u] : * \to \overline{\mathfrak{M}}_{g(u),1}$ corresponding to a stable curve F_u. The vertices of U are called *pointed vertices* and the stable curves F_u are *decorations*. An *automorphism* of a stable graph consists of two bijections of the sets $\mathrm{Vert}(\Gamma)$ and $H(\Gamma)$ compatible with σ, τ and the function g, fixing the legs pointwise, and sending pointed vertices to pointed vertices with identical decorations.

Given a stable graph Γ and an edge e we produce three new stable graphs. The graph $\Gamma \smallsetminus e$ is obtained by deleting the edge e (if the resulting graph contains a bivalent vertex of genus 0, then replace it by a single edge or half-edge, as appropriate). The graph $\Gamma | e$ is obtained by cutting e into two legs. The graph Γ / e is obtained by contracting the edge e; if the endpoints of e are two distinct vertices then one identifies them and adds their genera, and if the endpoints are the same then one increases the genus of that vertex by one. More generally, if K is a set of edges then we construct $\Gamma \smallsetminus K$, $\Gamma | K$, and Γ / K by iterating the above constructions.

Given a stable graph Γ, we define stacks

$$\begin{aligned}
\overline{\mathfrak{M}}(\Gamma) &:= \left(\prod_{v \in \mathrm{Vert}(\Gamma) \smallsetminus U} \overline{\mathfrak{M}}_{g(v),\mathrm{leg}(v)} \times \prod_{v \in U} * \right), \\
\overline{\mathfrak{M}}((\Gamma)) &:= \overline{\mathfrak{M}}(\Gamma) /\!\!/ \operatorname{Aut}(\Gamma).
\end{aligned}$$

The substacks $\mathfrak{M}(\Gamma) \subset \overline{\mathfrak{M}}(\Gamma)$ and $\mathfrak{M}((\Gamma)) \subset \overline{\mathfrak{M}}((\Gamma))$ are defined analogously. Observe that as Γ runs over the isomorphism classes of graphs (of type (g, n)) with no pointed vertices, $\mathfrak{M}(\Gamma)$ runs over the open strata of $\overline{\mathfrak{M}}_{g,n}$. The decorations $[F_u]$ on the pointed univalent vertices define an immersion $\overline{\mathfrak{M}}(\Gamma) \to \overline{\mathfrak{M}}(\Gamma')$, where Γ' is obtained from Γ by replacing all pointed vertices by ordinary vertices. Note that there is a canonical isomorphism $\overline{\mathfrak{M}}((\Gamma|K)) \cong \overline{\mathfrak{M}}(\Gamma) /\!\!/ \operatorname{Stab}(K)$.

An edge e determines a gluing morphism $\overline{\mathfrak{M}}(\Gamma) \to \overline{\mathfrak{M}}(\Gamma/e)$ as follows. Let v, v' be the endpoints of e. If v and v' are distinct then this morphism is induced by the gluing

morphism $\overline{\mathfrak{M}}_{g(v),\tau^{-1}(v)} \times \overline{\mathfrak{M}}_{g(v'),\tau^{-1}(v')} \longrightarrow \overline{\mathfrak{M}}_{g(v)+g(v'),\tau^{-1}(v\sqcup v')\smallsetminus e}$ defined by gluing curves together at the marked points corresponding to the half-edges of e. If $v = v'$ then it is induced by the gluing morphism $\overline{\mathfrak{M}}_{g(v),\tau^{-1}(v)} \longrightarrow \overline{\mathfrak{M}}_{g(v)+1,\tau^{-1}(v)\smallsetminus e}$ again defined by gluing marked points together to form a node. More generally, a set K of edges determines a gluing morphism

$$\widetilde{\xi}_K : \overline{\mathfrak{M}}(\Gamma) \to \overline{\mathfrak{M}}(\Gamma/K),$$

and if K is $\mathrm{Aut}(\Gamma)$-invariant then this morphism is equivariant and hence descends to a morphism

$$\xi_K : \overline{\mathfrak{M}}((\Gamma)) \to \overline{\mathfrak{M}}((\Gamma/K)).$$

Let $*_{g,n}$ denote the stable graph consisting of a single genus g vertex and n legs; its automorphism group is trivial. Contracting all edges of Γ gives a map $\xi_{E(\Gamma)} : \overline{\mathfrak{M}}((\Gamma)) \to \mathfrak{M}((*_{g,n})) = \overline{\mathfrak{M}}_{g,n}$. The restriction of $\xi_{E(\Gamma)}$ to $\mathfrak{M}(\Gamma)$ is the inclusion of the open stratum labeled by Γ.

Given a leg h, there is a forgetful morphism $\overline{\mathfrak{M}}(\Gamma) \to \overline{\mathfrak{M}}(\Gamma \smallsetminus h)$ given by forgetting the marked point on a stable curve corresponding to the leg h. (As usual, if we forget a marked point on a genus zero component with only two additional marked points or nodes then we collapse that component to a node). More generally, given a set of edges K, there is a forgetful morphism

$$\widetilde{\pi}_K : \overline{\mathfrak{M}}(\Gamma) \to \overline{\mathfrak{M}}(\Gamma \smallsetminus K);$$

if K is $\mathrm{Aut}(\Gamma)$-invariant, then it descends to

$$\pi_K : \overline{\mathfrak{M}}((\Gamma)) \to \overline{\mathfrak{M}}((\Gamma \smallsetminus K)).$$

4.3. **Vector bundles on $\overline{\mathfrak{M}}(\Gamma)$ and stripping and splitting.** On $\overline{\mathfrak{M}}_{g,n}$ there are complex line bundles $\widetilde{L}_1, \ldots, \widetilde{L}_n$ whose fibres at a given curve are the tangent spaces at each of the marked points. Hence a half-edge h of Γ determines a complex line bundle $\widetilde{L}_h$ on $\overline{\mathfrak{M}}(\Gamma)$. More generally, a set H of half-edges determines a vector bundle $\widetilde{L}_H = \oplus_{h\in H}\widetilde{L}_h$, and this bundle is $\mathrm{Aut}(\Gamma)$-equivariant whenever H is $\mathrm{Aut}(\Gamma)$-invariant. Assuming H is invariant, $\widetilde{L}_H$ is classified by a map $\widetilde{L}_H : \overline{\mathfrak{M}}(\Gamma) \to BU(1)^H$ which descends to homotopy orbits to give a map

$$\begin{aligned} L_H : \overline{\mathfrak{M}}((\Gamma)) &\to B(U(1)^H \rtimes \mathrm{Aut}(H,\Gamma)) \\ &= E\,\mathrm{Aut}(H,\Gamma) \times_{\mathrm{Aut}(H,\Gamma)} BU(1)^H, \end{aligned}$$

where $\mathrm{Aut}(H,\Gamma)$ is the group of permutations of H which are induced by automorphisms of Γ.

Given a set H of half-edges, let E_H denote the set of edges which contain elements of H. The following proposition will be used in the proof of Theorem 5.1.

Proposition 4.2. *(i) If H a set of half-edges consisting of an $\mathrm{Aut}(\Gamma)$-orbit of edges between ordinary vertices (not necessarily distinct, having genera g_1 and g_2) then the $\mathrm{Aut}(\Gamma)$-equivariant map*

$$\mathfrak{M}(\Gamma) \xrightarrow{\widetilde{\pi}_{E_H} \times \widetilde{L}_H} \mathfrak{M}(\Gamma \smallsetminus E_H) \times BU(1)^H \tag{4.3}$$

is a homology isomorphism in degrees $* \leq \min\{g_1/2-1, g_2/2-1\}$, and hence it induces an isomorphism in this range on the homotopy orbits.

(ii) If H is an $\mathrm{Aut}(\Gamma)$-orbit of half-edges h (incident at an ordinary vertex of genus g_1) such that $\sigma(h)$ is incident at a univalent pointed vertex then the $\mathrm{Aut}(\Gamma)$-equivariant map

$$\mathfrak{M}(\Gamma) \xrightarrow{\widetilde{\pi}_{E_H} \times \widetilde{L}_H} \mathfrak{M}(\Gamma \smallsetminus E_H) \times BU(1)^H, \tag{4.4}$$

is a homology isomorphism in degrees $* \leq g_1/2 - 1$, and hence it induces a homology isomorphism in this range on homotopy orbits.

Proof. Bödigheimer and Tillmann proved in [BT01] that Harer-Ivanov stability [Har85], [Iva93] implies that the $\Sigma_P \times \Sigma_Q$-equivariant "stripping-and-splitting" map

$$\mathfrak{M}_{g,P\sqcup Q} \xrightarrow{\pi_Q \times L_Q} \mathfrak{M}_{g,P} \times BU(1)^Q$$

is a homology isomorphism in degrees $* \leq g/2 - 1$. The proposition is a straighforward application of this. The statements about homotopy quotients follow from a standard argument with the Leray-Serre spectral sequence. □

Remark 4.5. Here is a proof of the theorem in [BT01], easier than the original one. The stripping and splitting map is the middle vertical arrow in the following diagram (whose rows are homotopy-fibrations)

$$\begin{tikzcd}
\mathfrak{M}_{g,P,\vec{Q}} \arrow[r] \arrow[d] & \mathfrak{M}_{g,P\sqcup Q} \arrow[r] \arrow[d] & BU(1)^Q \arrow[d, equal] \\
\mathfrak{M}_{g,P} \arrow[r] & \mathfrak{M}_{g,P} \times BU(1)^Q \arrow[r] & BU(1)^Q,
\end{tikzcd}$$

where $\mathfrak{M}_{g,P,\vec{Q}}$ is the moduli stack of smooth curves of genus g with $|P|$ marked points and $|Q|$ additional marked points equipped with a nonzero tangent vector. The left vertical arrow is a homology equivalence in the stable range by Harer-Ivanov stability. The base space is simply-connected, so both fibrations are simple. Thus a straightforward application of the Leray-Serre spectral sequence finishes the proof.

4.4. **The irreducible components of the boundary.** Let $\mathcal{D}$ denote the set of irreducible components of the boundary of $\overline{\mathfrak{M}}_{g,n}$. The elements of $\mathcal{D}$ are indexed by the (isomorphism classes of) stable graphs of genus g with n legs, a single edge $e = \{h_1, h_2\}$, and no pointed vertices. We call such a stable graph *elementary*. The elementary graph consisting of a single vertex with a loop and n legs is denoted Γ_{irr}, and it corresponds to the locus of curves with a non-separating node. The other boundary components correspond to elementary graphs with two vertices; they are indexed by the partition of g between the two vertices and the subset $P \subset \{1, \ldots, n\}$ of legs incident at the vertex of lesser genus.

Given $\alpha \in \mathcal{D}$, let Γ_α denote the corresponding elementary graph. The sole edge of Γ_α determines a gluing morphism

$$\xi_\alpha : \overline{\mathfrak{M}}((\Gamma_\alpha)) \to \overline{\mathfrak{M}}_{g,n}$$

which is a complex codimension 1 immersion whose image is precisely the boundary component α. The elementary graphs with two vertices and a nonzero number of legs incident at the vertex of smaller genus correspond to the boundary components which have no self-intersections, so the gluing morphisms for these are embeddings. We will only be interested in the self-intersecting boundary components. Let $\mathcal{D}^+ \subset \mathcal{D}$ denote the set of boundary components which have nontrivial self-intersections. See Figure 1.

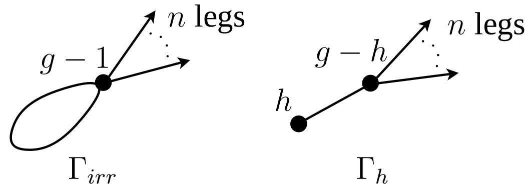

FIGURE 1. Elementary graphs indexing the boundary components of $\mathcal{D}^+$; i.e. those which have nontrivial self-intersections.

Let Γ_α be an elementary graph with edge $e = \{h_1, h_2\}$. The gluing morphism $\xi_\alpha = \xi_e : \overline{\mathfrak{M}}((\Gamma_\alpha)) \to \overline{\mathfrak{M}}_{g,n}$ has normal bundle is $L_{h_1} \otimes L_{h_2}$ (see [HM98] p.101). Note that if $\alpha = (irr)$ then there is an automorphism swapping the two half-edges, so one only has this tensor decomposition after pulling back to $\overline{\mathfrak{M}}(\Gamma_{irr})$. Thus the structure group of the normal bundle of ξ_α can be uniformly written as $T(2) \rtimes \mathrm{Aut}(\Gamma_\alpha)$.

5. THE PONTRJAGIN-THOM MAPS FOR $\overline{\mathfrak{M}}_{g,n}$ IN HOMOLOGY

5.1. **Statement of results and outline of proof.** In this section we will state and prove our main result in full generality. But first we need to set up some terminology and notation. Throughout this section we will drop the notational distinction between stacks and their homotopy types and between stack quotients and the corresponding homotopy quotients. We fix a genus g and number n of marked points throughout.

For any boundary component $\alpha \in \mathcal{D}^+$, the complex codimension 1 immersion $\xi_\alpha : \overline{\mathfrak{M}}((\Gamma_\alpha)) \to \overline{\mathfrak{M}}_{g,n}$ has a normal bundle $\nu(\alpha)$. By the discussion of section 4.4, the structure group of $\nu(\alpha)$ is equipped with a distinguished lift to $T(2) \rtimes \mathrm{Aut}(\Gamma_\alpha)$. Hence the Pontrjagin-Thom construction from Theorem 3.2 provides maps

$$\Phi^1_\alpha : \overline{\mathfrak{M}}_{g,n} \to Q\overline{\mathfrak{M}}(\Gamma_\alpha)^{\nu(\alpha)} \to QB(T(2) \rtimes \mathrm{Aut}(\Gamma_\alpha))^V,$$
$$\Phi^0_\alpha : \overline{\mathfrak{M}}_{g,n} \to Q\overline{\mathfrak{M}}(\Gamma_\alpha)^{\nu(\alpha)} \to QBU(1)^V,$$

(the difference is whether or not we use the lifted structure group). We will prove that a product of several maps of the above type is surjective on homology (with field coefficients) in a range of degrees. We consider both, Φ^1_α and Φ^0_α, because the target of the former has larger homology, while the range of surjectivity is often larger for the latter.

Fix a subset $A \subset \mathcal{D}^+$ and a function $\ell : A \to \{0, 1\}$, such that $\ell(irr) = 1$ if $(irr) \in A$. For each $\alpha \in A$, define

$$G_\alpha = \begin{cases} N(2) & \alpha = (irr) \\ T(2) & \alpha \neq (irr), \ell(\alpha) = 1 \\ U(1) & \ell(\alpha) = 0 \end{cases}$$

Thus the target of $\Phi^{\ell(\alpha)}_\alpha$ is QBG^V_α. Set $g_{irr} = 1$, and for any other elementary graph Γ_α set g_α to be the lesser of the genera of the two vertices. We define an A-*partition* of g to be a set $\vec{m} := (m_\alpha)_{\alpha\in A}$ of nonnegative integers such that $r := g - \sum_{\alpha\in A} m_\alpha g_\alpha$ is nonnegative. Given an A-partition $\vec{m}$, we set

$$\begin{aligned} c(A, \ell, \vec{m}) :=& \min \{ r/2 - 1, \quad m_\alpha/2 \quad (\alpha \in A), \\ & \qquad g_\alpha/2 - 1 \quad (\alpha \in A \text{ with } \ell(\alpha) = 1 \text{ and } \alpha \neq irr) \}, \\ c(A, \ell) :=& \max\{c(A, \ell, \vec{m}) \quad | \quad \vec{m} \text{ an } A\text{-partition of } g\}. \end{aligned}$$

Theorem 5.1. *Given a pair* $(A \subset \mathcal{D}^+, \ell : A \to \{0,1\})$*, the map*

$$\prod_{\alpha \in A} \Phi_\alpha^{\ell(\alpha)} : \mathrm{Ho}(\overline{\mathfrak{M}}_{g,n}) \to \prod_{\alpha \in A} QBG_\alpha^V$$

is surjective on ordinary homology with field coefficients in degrees $* \leq c(A, \ell)$.

Remark 5.2.

1. The special case $A = \{(irr)\}$ gives case (1) of Theorem 1.2. Taking $A = \{h\}$ and $\ell(h) = 1$ or 0 give the other two cases.
2. As we will see in the proof of this theorem, the homology surjectivity comes from boundary components that have high numbers of self-intersections. Thus Pontrjagin-Thom maps for the boundary components which are embedded (rather than immersed) factor as

$$\overline{\mathfrak{M}}_{g,n} \to BT(2)^V \to Q(BT(2)).$$

Such maps cannot be surjective in homology in a range because the second map is not. This is why the theorem refers only to self-intersecting boundary components.

Here is an outline of the proof of Theorem 5.1. We consider the following diagram:

$$\begin{tikzcd}
\mathfrak{M}((\Gamma)) \arrow[r, "\xi_{E(\Gamma)}"] \arrow[d, "L_{H_0}"] & \overline{\mathfrak{M}}_{g,n} \arrow[r, "\prod \Phi_\alpha^{\ell(\alpha)}"] & \prod_{\alpha \in A} QBG_\alpha^V \\
\prod_{\alpha \in A} B(G_\alpha \wr \Sigma_{m_\alpha}) \arrow[rr, "\prod \mathrm{gc}_\alpha"] & & \prod_{\alpha \in A} Q_{(m_\alpha)}(BG_\alpha)_+ . \arrow[u, "\prod Q\,\mathrm{inc}"']
\end{tikzcd} \tag{5.3}$$

Here $\mathfrak{M}((\Gamma))$ is the open stratum in $\overline{\mathfrak{M}}_{g,n}$ determined by a certain stable graph Γ. Going around counter-clockwise: the map L_{H_0} is the classifying map for a vector bundle determined by a certain set H_0 of half-edges of Γ; the maps gc_α are components of the group completion map appearing in the Barratt-Priddy-Quillen-Segal Theorem; the last map, $Q\,\mathrm{inc}$, is induced by the inclusion of the zero section into the Thom space. We will define each of these maps in detail and show that they are each surjective on homology in certain ranges of degrees, and that the above diagram commutes up to homotopy.

5.2. **Definition of the stable graph** Γ**.** Fix an A-partition $\vec{m}$ such that $c(A, \ell, \vec{m})$ is maximal and construct Γ as follows. There is a vertex v of genus r, n legs incident at v, and m_{irr} loops at v (if $(irr) \in A$). For each $\alpha \in A \smallsetminus (irr)$ there are m_α additional vertices of genus g_α, each of which is connected to v by a single edge. For each α with $\ell(\alpha) = 0$ the univalent vertices of genus g_α are pointed (all with the same decoration). See figure 2.

The automorphism group of Γ is

$$\mathrm{Aut}(\Gamma) \cong \prod_{\alpha \in A} \mathrm{Aut}(\Gamma_\alpha) \wr \Sigma_{m_\alpha}.$$

Let H_0 denote the set of all half-edges incident at ordinary vertices. One then has

$$U(1)^{H_0} \rtimes \mathrm{Aut}(\Gamma) \cong \prod_{\alpha \in A} G_\alpha \wr \Sigma_{m_\alpha}.$$

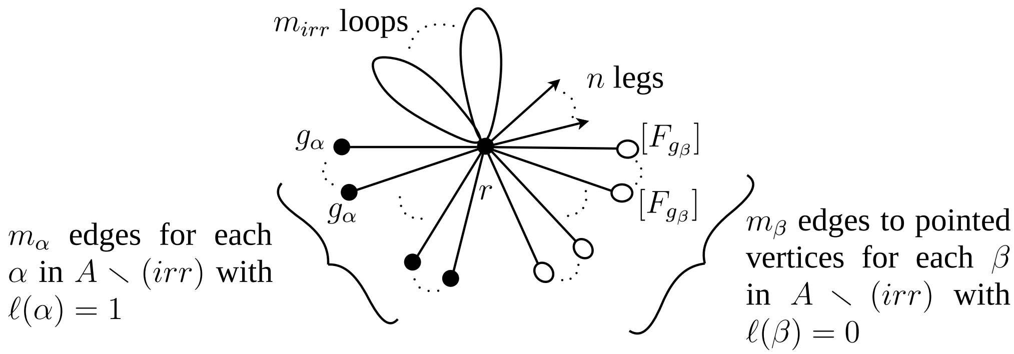

FIGURE 2. The test graph Γ for the proof of Theorem 5.1.

5.3. **The classifying map of L_{H_0}.** Recall that H_0 is the set of half-edges of Γ which are incident at ordinary vertices; this determines a vector bundle L_{H_0} on $\overline{\mathfrak{M}}((\Gamma))$ with structure group $U(1)^{H_0} \rtimes \mathrm{Aut}(\Gamma) = \prod_{\alpha \in A} G_\alpha \wr \Sigma_{m_\alpha}$.

Lemma 5.4. *The classifying map $L_{H_0} : \overline{\mathfrak{M}}((\Gamma)) \to \prod_{\alpha \in A} B(G_\alpha \wr \Sigma_{m_\alpha})$ is surjective on homology in degrees*

$$* \leq \min \{ r/2 - 1, \ \ g_\alpha/2 - 1 \ \ (\textit{for } \alpha \textit{ such that } \ell(\alpha) = 1) \} .$$

Proof. Let E_{irr} denote the set of loops at the central vertex, and for $\alpha \neq (irr)$ let E_α denote the set of edges of Γ which meet an outer vertex of genus g_α. Let H_α denote the set of half-edges lying in E_α which are incident at an ordinary vertex. There is an $\mathrm{Aut}(\Gamma)$-invariant decomposition $H_0 = \coprod_{\alpha \in A} H_\alpha$, The classifying map for the bundle $\widetilde{L}_{H_0}$ on $\overline{\mathfrak{M}}(\Gamma)$ factors as

$$\begin{aligned} \overline{\mathfrak{M}}(\Gamma) &\to \overline{\mathfrak{M}}(\Gamma \smallsetminus E_{\alpha_1}) \times BU(1)^{H_{\alpha_1}} \\ &\longrightarrow \overline{\mathfrak{M}}(\Gamma \smallsetminus E_{\alpha_1} \cup E_{\alpha_2}) \times BU(1)^{H_{\alpha_1} \cup H_{\alpha_2}} \\ &\to \cdots \to \overline{\mathfrak{M}}(\Gamma \smallsetminus \cup_{\alpha \in A} E_\alpha) \times BU(1)^{H_0} \\ &\xrightarrow{\text{proj}} BU(1)^{H_0}, \end{aligned}$$

where the first sequence of arrows are the stripping-and-splitting maps of Proposition 4.2, and the final arrow is simply projection. All of these maps are $\mathrm{Aut}(\Gamma)$-equivariant, and the final map admits an equivariant section by choosing a fixed point in $\overline{\mathfrak{M}}(\Gamma \smallsetminus \cup_\alpha E_\alpha)$. One obtains the classifying map of the bundle L_{H_0} on $\overline{\mathfrak{M}}((\Gamma))$ by passing to homotopy orbits. The sequence of stripping-and-splitting maps induce homology isomorphisms in the stated range of degrees on homotopy orbits and the projection induces a homology epimorphism on the homotopy orbits. □

5.4. **Symmetric groups and group completion.** We now discuss the map gc occurring in diagram (5.3). It is a special case of a general construction, called “group completion”. Let X be a connected space. There is an m-fold covering

$$E(\Sigma_{m-1} \times 1) \times_{\Sigma_{m-1} \times 1} X^m \to E\Sigma_m \times_{\Sigma_m} X^m$$

induced by the index m inclusion $\Sigma_{m-1} \times 1 \hookrightarrow \Sigma_m$. The Becker-Gottlieb transfer is a map of spectra

$$\mathrm{trf} : \Sigma^\infty (E\Sigma_m \times_{\Sigma_m} X^m)_+ \to \Sigma^\infty (E(\Sigma_{m-1} \times 1) \times_{\Sigma_{m-1} \times 1} X^m)_+$$

which can be described (when X is a manifold or local quotient stack) as the Pontrjagin-Thom construction for the covering projection. The adjoint of the transfer is $E\Sigma_m \times_{\Sigma_m} X^m \to Q_{(m)}(E(\Sigma_{m-1}\times 1)\times_{\Sigma_{m-1}\times 1} X^m)_+$, where $Q_{(m)}$ denotes the m^{th} component. The group completion map is then the composition

$$\mathrm{gc}_m : E\Sigma_m \times_{\Sigma_m} X^m \xrightarrow{\mathrm{trf}} Q(E(\Sigma_{m-1}\times 1)\times_{\Sigma_{m-1}\times 1} X^m)_+ \to Q_{(m)}X_+,$$

where the second map is induced by projecting onto the m^{th} component of X^m.

The name stems from the following. One can put a monoid structure on the union

$$\coprod_m E\Sigma_m \times_{\Sigma_m} X^m,$$

and the maps $\{\mathrm{gc}_m\}$ assemble to a monoid map $\mathrm{gc} : \coprod_m E\Sigma_m \times_{\Sigma_m} X^m \to QX_+$. The Barratt-Priddy-Quillen-Segal Theorem (see e.g. [Seg74]) asserts that this map is the homotopy-theoretic group completion of the above monoid.

Lemma 5.5. *For any connected space X, the map $\mathrm{gc}_m : E\Sigma_m \times_{\Sigma_m} X^m \to Q_{(m)}X_+$ induces a isomorphisms in homology with field coefficients*[2] *in degrees $* \leq (m-1)/2$.*

Proof. This is a well-known consequence of homology stability for symmetric groups (with twisted coefficients). After choosing a basepoint in X one has stabilization maps

$$j_m : E\Sigma_m \times_{\Sigma_m} X^m \to E\Sigma_{m+1} \times_{\Sigma_{m+1}} X^{m+1}$$

whose colimit is denoted by $E\Sigma_\infty \times_{\Sigma_\infty} X^\infty$. The induced map

$$\mathrm{gc}_\infty : E\Sigma_\infty \times_{\Sigma_\infty} X^\infty \to QX_+$$

is a homology isomorphism onto the basepoint component by the group completion theorem (see e.g. [MS76], [Seg73]). Up to a shift of component the stabilization map from $E\Sigma_m \times_{\Sigma_m} X^m$ to the colimit followed by gc_∞ agrees with gc_m. The stabilization map j_m induces isomorphism in homology in degrees $* \leq m/2 - 1$, and epimorphism when $* \leq m/2$.[3] □

When $X = BG_\alpha$ and $m = m_\alpha$, the map gc_m of this lemma is precisely the map gc_α occuring in Lemma 5.3, so the map

$$\prod \mathrm{gc}_\alpha : \prod_{\alpha\in A} B(G_\alpha \wr \Sigma_{m_\alpha}) \to Q_{(m_\alpha)}(BG_\alpha)_+$$

is a homology epimorphism in degrees $* \leq \min\{m_\alpha/2 \mid \alpha \in A\}$.

[2]If the homology of X is of finite type, then the coefficients can be arbitrary.

[3]This has probably been known for a long time. One proof can be found in [Han07], based on a result of [Bet02]. Another proof is a combination of Proposition 1.6 in [HW07] with the main result of [KT76]. The authors are not aware of a previously published proof.

5.5. **Homology of the Thom spectra and their infinite loop spaces.** The third arrow, $\prod Q\,\mathrm{inc} : \prod_{\alpha\in A} Q_{(m_\alpha)}(BG_\alpha)_+ \to \prod_{\alpha\in A} Q(BG_\alpha^V)$, in the counterclockwise composition in the diagram (5.3) is induced by the inclusion of the zero section of the vector bundle V over BG_α. We show here that it is surjective in homology with field coefficients. This is the only part of the proof of Theorem 5.1 where field coefficients are used seriously. The homology surjectivity is immediate from the following two lemmata.

Lemma 5.6. *The inclusions of the zero sections*

$$BT(2) \hookrightarrow BT(2)^V,$$
$$BN(2) \hookrightarrow BN(2)^V,$$
$$BU(1) \hookrightarrow BU(1)^V$$

induce surjections in homology with field coefficients.

Lemma 5.7. *If $f : X \to Y$ is a pointed map between pointed spaces which is surjective in homology with coefficients in a field $\mathbb{F}$, then the induced map $Qf : QX \to QY$ is surjective on homology with coefficients in $\mathbb{F}$.*

Lemma 5.7 is a well-known fact which we discuss in section 5.7.

Proof of Lemma 5.6. Let ι denote one of the three above inclusions. We shall prove the equivalent statement that ι^* is injective on cohomology. The composition of the Thom isomorphism followed by ι^* is equal to multiplication by the Euler class $e(V)$ of V, so it suffices to show that $e(V)$ is not a zero-divisor in each of the three cases. For $BT(2)$, $BU(1)$ and $BN(2); \mathrm{char}(\mathbb{F}) \neq 2$, the computation is easy and well-known.

For $BN(2)$ and $\mathrm{char}(\mathbb{F}) = 2$, we argue as follows. The homogeneous space $U(2)/N(2)$ is diffeomorphic to $\mathbb{RP}^2$, so there is a fibration

$$\mathbb{RP}^2 \to BN(2) \xrightarrow{p} BU(2),$$

which is simple because $BU(2)$ is simply-connected. Put $y_i := p^* c_i \in H^*(BN(2); \mathbb{F})$. Consider the Leray-Serre spectral sequence for the fibration. The cohomology of the real projective plane is

$$H^*(\mathbb{RP}^2; \mathbb{F}) \cong \mathbb{F}[w]/(w^3),$$

where $w \in H^1(\mathbb{RP}^2; \mathbb{F})$. One has $H^1(BN(2); \mathbb{F}_2) \cong \mathbb{F}_2$, since $\pi_1 BN(2) \cong \mathbb{Z}/2$, and hence the spectral sequence collapses. Thus for $\mathrm{char}\,\mathbb{F} = 2$,

$$H^*(BN(2); \mathbb{F}) \cong \mathbb{F}[w, y_1, y_2]/(w^3).$$

The line bundle $V \to BN(2)$ is the tensor product of the determinant line bundle $BN(2) \to BU(2) \stackrel{B\det}{\to} BS^1$ with the signum line bundle $BN(2) \to B\mathbb{Z}/2 \stackrel{B\,\mathrm{inc}}{\to} BS^1$ and thus $c_1(V) = y_1 + w^2$, which is not a zero divisor in $\mathbb{F}[w, y_1, y_2]/(w^3)$. □

5.6. **Homotopy commutativity of diagram** (5.3)**.** We first establish a general fact about Pontrjagin-Thom maps. We say that two maps are *weakly homotopic* if their restrictions to any compact subset of the domain are homotopic. Weakly homotopic maps induce identical homomorphisms on homotopy groups and in any generalized homology theory.

Lemma 5.8. *Let $D \stackrel{f}{\looparrowright} M \stackrel{g}{\leftarrow} Z$ be a diagram of smooth manifolds such that*

(1) *f is a proper immersion,*

(2) $\mathrm{Im}(g) \subset \mathrm{Im}(f)$,
(3) *the projection* $q : D \times_M Z \to Z$ *is a finite sheeted covering.*

Let h *be the projection* $D \times_M Z \to M$ *and* $\mathrm{inc} : D \to D^{\nu(f)}$ *the zero section. Then the diagram of spaces*

(5.9)
$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "\mathrm{trf}_q"] & M \arrow[r, "\mathrm{PT}_f"] & QD^{\nu(f)} \\ Q(D \times_M Z)_+ \arrow[rr, "Qh"] & & QD_+ \arrow[u, "Q\,\mathrm{inc}"'] \end{tikzcd}$$

commutes up to homotopy. If D, M *and* Z *are differentiable stacks then the same statement is true except that the diagram is only weakly homotopy commutative.*

Proof. Choose a map $j : D \to \mathbb{R}^n$ such that $(f, j) : D \to M \times \mathbb{R}^n$ is an embedding; it is proper because f is proper. The map

$$\begin{aligned} k :& D \times_M Z \hookrightarrow Z \times \mathbb{R}^n \\ & x \mapsto (q(x), j \circ h(x)) \end{aligned}$$

is therefore also a proper embedding. Identifying $\nu((f, j))$ with a tubular neighborhood of D in $M \times \mathbb{R}^n$, the inverse image under $g \times \mathrm{id}$ of this neighborhood is a tubular neighborhood of $D \times_M Z$ in $Z \times \mathbb{R}^n$ which we identify with the normal bundle $\nu(k)$. There are canonical identifications

$$\begin{aligned} \nu(k) &\cong (D \times_M Z) \times \mathbb{R}^n, \\ \nu((f, j)) &\cong \nu(f) \oplus \mathbb{R}^n, \end{aligned}$$

and the projection $q : D \times_M Z \to D$ identifies the trivial factor of $\nu((f, j))$ with $\nu(k)$. See Figure 5.6. Using these choices it is easy to check that either composition in (5.9) sends $(z, x) \in Z \times \mathbb{R}^n$ to ∞ if it is not in the tubular neighborhood of $D \times_M Z$ in $Z \times \mathbb{R}^n$; and if (z, x) does lie in the tubular neighborhood of $D \times_M Z$, corresponding to a point $((d, z), y) \in \nu(k)$, then it is sent to $((d, 0), y) \in \nu(f) \oplus \mathbb{R}^n$. Letting $n \to \infty$ and taking the adjoint diagram completes the proof in the case of manifolds. The statement for stacks follows from this, because any homotopy class of maps $K \to \mathrm{Ho}(Z)$, K a finite CW complex, can be represented by a submersion $K' \to Z$ (where K' is a manifold homotopy equivalent to K). □

Lemma 5.10. *The diagram* (5.3) *is weakly homotopy commutative.*

Proof. For each $\alpha \in A$ one sees that the morphisms

$$\overline{\mathfrak{M}}((\Gamma_\alpha)) \looparrowright \overline{\mathfrak{M}}_{g,n} \leftarrow \mathfrak{M}((\Gamma))$$

satisfy the hypotheses of Lemma 5.8. One easily verifies that

$$\overline{\mathfrak{M}}((\Gamma_\alpha)) \times_{\overline{\mathfrak{M}}_{g,n}} \mathfrak{M}((\Gamma)) \cong \mathfrak{M}((\Gamma|e)),$$

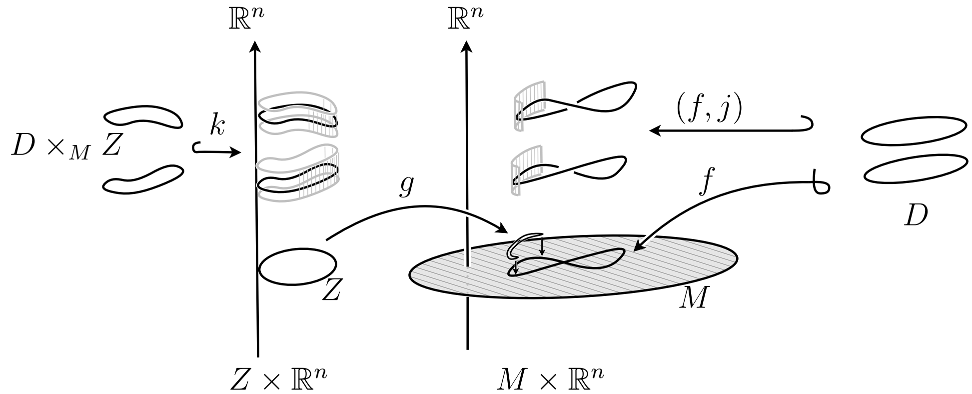

FIGURE 3. Illustration of geometry of Lemma 5.8. The normal bundle of D canonically splits as $\nu(f) \oplus \mathbb{R}^n$; the trivial factor and its pullback to $D \times_M Z$ are shown in grey.

where e is an edge of the type specified by α. We denote the projection onto the first factor by p. Now consider the following diagram:

$$\begin{tikzcd}
B(G_\alpha \wr \Sigma_{m_\alpha}) \arrow[d, "\mathrm{trf}"] & \mathfrak{M}((\Gamma)) \arrow[l] \arrow[d, "\mathrm{trf}"'] \arrow[dr, bend left, "\xi_{E(\Gamma)}"] & \\
QB(G_\alpha \wr (\Sigma_{m_\alpha - 1} \times 1))_+ \arrow[d, "Q\,\mathrm{proj}"] & Q\mathfrak{M}((\Gamma|e))_+ \arrow[l] \arrow[d, "Qp"'] & \overline{\mathfrak{M}}_{g,n} \arrow[ddl, bend left, "\mathrm{PT}_{\xi_\alpha}"] \\
Q(BG_\alpha)_+ \arrow[d, "Q\,\mathrm{inc}"] & Q\overline{\mathfrak{M}}((\Gamma_\alpha))_+ \arrow[l] \arrow[d, "Q\,\mathrm{inc}"'] & \\
Q_m BG_\alpha^V & Q\overline{\mathfrak{M}}((\Gamma_\alpha))^{\nu(\alpha)} \arrow[l] &
\end{tikzcd}$$

The right triangle is homotopy commutative by Lemma 5.8. The horizontal maps are all (induced by) classifying maps of suitable vector bundles. It is clear that the left part of the diagram is also commutative and that the composition along the left side is precisely the α component of the counterclockwise composition in Lemma 5.3. $\square$

5.7. **A quick review of homology of infinite loop spaces.** We recall the description of the homology of the free infinite loop space QX with coefficients in a field $\mathbb{F}$. In characteristic zero the description is easy and classical; in finite characteristic the standard reference is [May76].

If $\mathbb{F}$ is a field, V a graded $\mathbb{F}$-vector space, then we denote by $\Lambda(V)$ the free graded-commutative $\mathbb{F}$-algebra generated by V.

Let X be a pointed space. There is a natural map $X \to Q(X)$, adjoint to the identity on $\Sigma^\infty X$ and thus a homomorphism $H_*(X) \to H_*(QX)$. Because QX is a homotopy-commutative H-space, the Pontrjagin product defines the structure of a graded-commutative $\mathbb{F}$-algebra on the homology $H_*(QX;\mathbb{F})$. Thus we obtain a ring homomorphism $\Lambda(\widetilde{H}_*(X;\mathbb{F})) \to H_*(QX;\mathbb{F})$. If $\mathrm{char}(\mathbb{F}) = 0$ then this is an isomorphism. This is a standard result of algebraic topology, see [MM65], p. 262 f.

If $\operatorname{char}\mathbb{F} = p > 0$ then the homology $H_*(QX;\mathbb{F})$ is much richer. The homology algebra $H_*(QX;\mathbb{F})$ is a module over an algebra of homology operations known as the *Dyer-Lashof operations* (they are also known as Araki-Kudo operations if $p = 2$). These operations measure the failure of chain-level commutativity of the Pontrjagin product.

For $p \neq 2$, these operations are:

$$\beta^\epsilon Q^s : H_n(QX;\mathbb{F}) \to H_{n+2s(p-1)-\epsilon}(QX;\mathbb{F})$$

for $\epsilon \in \{0,1\}$ and $s \in \mathbb{Z}_{\geq \epsilon}$. Given a sequence $I = (\epsilon_1, s_1, \ldots, \epsilon_n, s_n)$, I is *admissible* if $s_{i+1} \leq ps_i - \epsilon_i$ for $i = 1, \ldots, n-1$. One defines the *excess*

$$e(I) = 2s_1 - \epsilon_1 - \sum_{i=2}^{n}(2s_i(p-1) - \epsilon_i)$$

and $b(I) = \epsilon_1$. Such a sequence determines an iteration of operations which is written Q^I.

When $p = 2$ the operations are of the form

$$Q^s : H_n(QX;\mathbb{F}) \to H_{n+s}(QX;\mathbb{F})$$

for $s \in \mathbb{Z}_{\geq 0}$. A sequence $I = (s_1, \ldots, s_n)$ is *admissible* if $s_{i+1} \leq 2s_i$ for each $i = 1, \ldots, n-1$. The excess is defined to be $e(I) = s_1 - \sum_{i=2}^n s_i$, and for convenience one puts $b(I) = 0$.

Let V be a graded $\mathbb{F}$-vector space and let B be a homogeneous basis of V. The *free unstable Dyer-Lashof module generated by* V is the $\mathbb{F}$-vector space $\mathrm{DL}_{\mathbb{F}}(V)$ on the basis

$$\{Q^I x \mid x \in B,\ I \text{ admissible}, e(I) + b(I) \geq \deg(x)\}.$$

Because $H_*(QX;\mathbb{F})$ has Dyer-Lashof operations, there is a ring homomorphism, compatible with the Dyer-Lashof operations

$$\Lambda(\mathrm{DL}_{\mathbb{F}}(\widetilde{H}_*(X;\mathbb{F}))) \to H_*(QX;\mathbb{F}),$$

and it is proven in [May76] that this is an isomorphism. This calculation immediately implies Lemma 5.7.

6. Comparison to the tautological algebra

Here we explain the relationship between the rational cohomology classes detected via Theorem 5.1 and the tautological algebra of $\overline{\mathfrak{M}}_{g,n}$.

Proposition 6.1. *The image of the homomorphism*

$$\Phi^*_{irr} : H^*(QBN(2)^V;\mathbb{Q}) \to H^*(\overline{\mathfrak{M}}_{g,n};\mathbb{Q})$$

is contained in the cohomology tautological algebra $\mathcal{R}^*(\overline{\mathfrak{M}}_{g,n})$. The analogous statement is true for the other maps studied in Theorem 1.2 and Theorem 5.1.

Before we can explain the definition of $\mathcal{R}^*(\overline{\mathfrak{M}}_{g,n})$ and the proof of 6.1, we need to say a few words about umkehr maps (also called “pushforward” or “Gysin map”) in cohomology and their relation to the Pontrjagin-Thom construction.

Let $f : M \to N$ be a proper smooth map between manifolds (or a proper representable morphism between differentiable local quotient stacks) of codimension d, and let $\mathrm{PT}_f : \Sigma^\infty N_+ \to \mathbb{T}\mathbf{h}(\nu(f))$ be its Pontrjagin-Thom map. A *cohomological orientation* of f is by

definition a Thom class in $H^d(\mathbb{T}\mathbf{h}(\nu(f))$. This orientation induces a *Thom isomorphism* $\mathrm{th} : H^*(M) \to H^{*+d}(M^{\nu(f)})$ (see [Rud98], ch. V for details). The *umkehr map* $f_!$ is defined as the composition

$$H^*(M) \cong H^*(\Sigma^\infty M_+) \xrightarrow{\mathrm{th}} H^{*+d}(M^{\nu(f)}) \tag{6.2}$$

$$\xrightarrow{\mathrm{PT}_f^*} H^{*+d}(\Sigma^\infty N_+) \cong H^{*+d}(N). \tag{6.3}$$

The tautological algebra has been studied by many authors; we refer to the survey paper [Vak06]. Here is the definition. One considers all natural morphisms $\overline{\mathfrak{M}}_{g,n+1} \to \overline{\mathfrak{M}}_{g,n}$ (forget the last point and collapse an unstable component if it shows up), $\overline{\mathfrak{M}}_{g-1,n+2} \to \overline{\mathfrak{M}}_{g,n}$, $\overline{\mathfrak{M}}_{h,k+1} \times \overline{\mathfrak{M}}_{g-h,n-k+1} \to \overline{\mathfrak{M}}_{g,n}$ (the gluing morphisms) and $\overline{\mathfrak{M}}_{g,n} \to \overline{\mathfrak{M}}_{g,n}$ (given by a permutation of the labelling set $\{1, \ldots, n\}$). All these morphisms are representable morphisms of complex-analytic stacks and so they have canonical orientations. Thus there are umkehr maps in integral cohomology for these morphisms. There is another, more traditional way to define the umkehr maps for complex orbifolds, based on rational Poincaré duality for the coarse moduli spaces, but this only works in rational cohomology.

Definition 6.4. The collection of tautological algebras

$$\mathcal{R}^*(\overline{\mathfrak{M}}_{g,n}) \subset H^{2*}(\overline{\mathfrak{M}}_{g,n}; \mathbb{Q})$$

is the smallest system of unital $\mathbb{Q}$-subalgebras which contain all classes $\psi_i = c_1(L_i) \in H^2(\overline{\mathfrak{M}}_{g,n}; \mathbb{Q})$, for all g, n and $i = 1, \ldots, n$ and which is closed under pushforward by the natural morphisms above.

We prove Proposition 6.1 only for the map $\Phi_{irr} : \overline{\mathfrak{M}}_{g,n} \to QBN(2)^V$, which is sufficient to clarify the pattern.

First recall that $H^*(QBN(2)^V; \mathbb{Q}) = \mathbb{Q}[a_{i,j}]$, where

$$a_{i,j} = \mathrm{th}(y_1^i y_2^j) \in H^{2+2i+4j}(BN(2)^V),$$

and y_i is the ith Chern class of the 2-dimensional complex vector bundle on $BN(2)$ induced by the inclusion $N(2) \to U(2)$. Thus we need to argue that $\Phi_{irr}^*(\mathrm{th}(y_1^i y_2^j))$ is in the tautological algebra. By the definition of Φ_{irr}, this is nothing else than $\mathrm{PT}_{\xi_{irr}}^*(\mathrm{th}(c_1(W)^i c_2(W)^j))$, where $W = L_{n+1} \oplus L_{n+2} \to \overline{\mathfrak{M}}_{g-1,n+(2)}$ is the sum of the natural line bundles (which is well-defined, although the last two points are permuted). This can be rewritten, using the definition of the umkehr map, as

$$(\xi_{irr})_!(c_1(W)^i c_2(W)^j) = (\xi_{irr})_!((\psi_{n+1} + \psi_{n+2})^i (\psi_{n+1}\psi_{n+2})^j).$$

This obviously lies in the tautological ring. There is a little argument needed, because we used the PT-map starting from $\overline{\mathfrak{M}}_{g-1,n+(2)}$, while the tautological algebra is defined using the map from the 2-fold cover $\overline{\mathfrak{M}}_{g-1,n+2}$. We leave this to the reader.

References

[Beh04] K. Behrend, *Cohomology of stacks*, Intersection theory and moduli, ICTP Lect. Notes, XIX, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004, pp. 249–294.

[Ber81] L. Bers, *Finite-dimensional Teichmüller spaces and generalizations*, Bull. Amer. Math. Soc. (N.S.) **5** (1981), no. 2, 131–172.

[Bet02] S. Betley, *Twisted homology of symmetric groups*, Proc. Amer. Math. Soc. **130** (2002), no. 12, 3439–3445 (electronic).

[BT01] C.-F. Bödigheimer and U. Tillmann, *Stripping and splitting decorated mapping class groups*, Cohomological methods in homotopy theory (Bellaterra, 1998), Progr. Math., vol. 196, Birkhäuser, Basel, 2001, pp. 47–57.

[BtD95] T. Bröcker and T. tom Dieck, *Representations of compact Lie groups*, Springer-Verlag, New York, 1995.

[DM69] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 75–109.

[Edi00] D. Edidin, *Notes on the construction of the moduli space of curves*, Recent progress in intersection theory (Bologna, 1997), Trends Math., Birkhäuser Boston, Boston, MA, 2000, pp. 85–113.

[FHT] D. Freed, M. Hopkins, and C. Teleman, *Loop groups and twisted K-theory I*, preprint, arXiv:0711.1906, 2007.

[Gal04] S. Galatius, *Mod p homology of the stable mapping class group*, Topology **43** (2004), no. 5, 1105–1132.

[GE06] S. Galatius and Y. Eliashberg, *Homotopy theory of compactified moduli spaces*, Oberwolfach Reports (2006), 761–766.

[GK98] E. Getzler and M. M. Kapranov, *Modular operads*, Compositio Math. **110** (1998), no. 1, 65–126.

[Hae84] A. Haefliger, *Groupoïdes d'holonomie et classifiants*, Astérisque (1984), no. 116, 70–97, Transversal structure of foliations (Toulouse, 1982).

[Han07] L. Hanbury, *Homology stability of mapping class groups and open-closed cobordism categories*, Ph.D. thesis, Oxford, 2007.

[Har85] J. L. Harer, *Stability of the homology of the mapping class groups of orientable surfaces*, Ann. of Math. **121** (1985), no. 2, 215–249.

[Hei05] J. Heinloth, *Somes notes on differentiable stacks*, Mathematisches Institut, Seminars 2004/05, Universität Göttingen (2005), 1–32.

[HM98] J. Harris and I. Morrison, *Moduli of curves*, Graduate Texts in Mathematics, vol. 187, Springer-Verlag, New York, 1998.

[HW07] A. Hatcher and N. Wahl, *Stabilization for mapping class groups of 3-manifolds*, preprint, ArXiv:math/0709.2173, 2007.

[Iva93] N. V. Ivanov, *On the homology stability for Teichmüller modular groups: closed surfaces and twisted coefficients*, Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991), Contemp. Math., vol. 150, Amer. Math. Soc., Providence, RI, 1993, pp. 149–194.

[Knu83] F. Knudsen, *The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$*, Math. Scand. **52** (1983), no. 2, 161–199.

[KT76] D. M. Kan and W. P. Thurston, *Every connected space has the homology of a $K(\pi, 1)$*, Topology **15** (1976), no. 3, 253–258.

[LMB00] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 39, Springer-Verlag, Berlin, 2000.

[May76] J. P. May, *The homology of E_∞-spaces*, The homology of iterated loop spaces, Springer-Verlag, Berlin, 1976, Lecture Notes in Mathematics, Vol. 533.

[MM65] J. W. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Ann. of Math. (2) **81** (1965), 211–264.

[Moe02] I. Moerdijk, *Orbifolds as groupoids: an introduction*, Orbifolds in mathematics and physics (Madison, WI, 2001), Contemp. Math., vol. 310, Amer. Math. Soc., Providence, RI, 2002, pp. 205–222.

[Mos57] G. D. Mostow, *Equivariant embeddings in Euclidean space*, Ann. of Math. **65** (1957), 432–446.

[MS76] D. McDuff and G. Segal, *Homology fibrations and the "group-completion" theorem*, Invent. Math. **31** (1975/76), no. 3, 279–284.

[MW07] I. Madsen and M. Weiss, *The stable moduli space of Riemann surfaces: Mumford's conjecture*, Ann. of Math. **165** (2007), 843–941.

[Noo05a] B. Noohi, *Cohomology of pretopological stacks*, preprint, available at http://www.math.fsu.edu/∼noohi/papers/papers.html, 2005.

[Noo05b] ———, *Foundations of topological stacks I*, preprint, ArXiv:math/0503247, 2005.

[RS06] J. W. Robbin and D. A. Salamon, *A construction of the Deligne-Mumford orbifold*, J. Eur. Math. Soc. (JEMS) **8** (2006), no. 4, 611–699.
[Rud98] Y. B. Rudyak, *On Thom spectra, orientability, and cobordism*, Springer-Verlag, 1998.
[Seg68] G. Segal, *Classifying spaces and spectral sequences*, Inst. Hautes Études Sci. Publ. Math. **34** (1968), 105–112.
[Seg73] ———, *Configuration-spaces and iterated loop-spaces*, Invent. Math. **21** (1973), 213–221.
[Seg74] ———, *Categories and cohomology theories*, Topology **13** (1974), 293–312.
[Ste67] N. E. Steenrod, *A convenient category of topological spaces*, Michigan Math. J. **14** (1967), 133–152.
[Vak06] R. Vakil, *The moduli space of curves and Gromov-Witten theory*, preprint, ArXiv:math.AG/0602347, 2006.
[Zun06] N. T. Zung, *Proper groupoids and momentum maps: linearization, affinity, and convexity*, Ann. Sci. École Norm. Sup. (4) **39** (2006), 841–869.